

Three-coloring graphs with no induced seven-vertex path I : the triangle-free case

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Abstract

In this paper, we give a polynomial time algorithm which determines if a given triangle-free graph with no induced seven-vertex path is 3-colorable, and gives an explicit coloring if one exists.

1 Introduction

We start with some definitions. All graphs in this paper are finite and simple. Let G be a graph and X be a subset of $V(G)$. We denote by $G[X]$ *the subgraph of G induced by X* , that is, the subgraph of G with vertex set X such that two vertices are adjacent in $G[X]$ if and only if they are adjacent in G . We denote by $G \setminus X$ the graph $G[V(G) \setminus X]$. If $X = \{v\}$ for some $v \in V(G)$, we write $G \setminus v$ instead of $G \setminus \{v\}$. Let H be a graph. If G has no induced subgraph isomorphic to H , then we say that G is *H -free*. For a family \mathcal{F} of graphs, we say that G is *\mathcal{F} -free* if G is F -free for every $F \in \mathcal{F}$. If G is not H -free, then G *contains H* . If $G[X]$ is isomorphic to H , then we say that X *is an H in G* .

For $n \geq 0$, we denote by P_{n+1} *the path with $n + 1$ vertices and length n* , that is, the graph with distinct vertices $\{p_0, p_1, \dots, p_n\}$ such that p_i is adjacent to p_j if and only if $|i - j| = 1$. For $n \geq 3$, we denote by C_n *the cycle of length n* , that is, the graph with distinct vertices $\{c_1, \dots, c_n\}$ such that c_i is adjacent to c_j if and only if $|i - j| = 1$ or $n - 1$. By convention, when explicitly describing a path or a cycle, we always list the vertices in order. Let G be a graph. When $G[\{p_0, p_1, \dots, p_n\}]$ is the path P_{n+1} , we say that $p_0 - p_1 - \dots - p_n$ *is a P_{n+1} in G* . Similarly, when $G[\{c_1, c_2, \dots, c_n\}]$ is the cycle C_n , we say that $c_1 - c_2 - \dots - c_n - c_1$ *is a C_n in G* . For $n \geq 3$, an *n -gon* in a graph G is an induced subgraph of G isomorphic to C_n . We also refer to a cycle of length three as a

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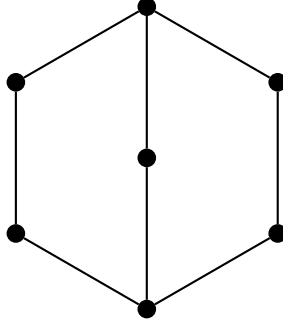


Figure 1: The shell.

triangle. Lastly, suppose C is a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We say that (C, p) , or $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$ with p , is a *shell* in G , provided that $p \in V(G) \setminus \{v_0, \dots, v_5\}$ is such that $N(p) \cap \{v_0, \dots, v_5\} = \{v_\ell, v_{\ell+3}\}$ for some $\ell \in \{0, 1, 2\}$. The shell is drawn in Figure 1.

A k -coloring of a graph G is a mapping $c : V(G) \rightarrow \{1, \dots, k\}$ such that if $x, y \in V(G)$ are adjacent, then $c(x) \neq c(y)$. If a k -coloring exists for a graph G , we say that the G is k -colorable. The COLORING problem is determining the smallest integer k such that a given graph is k -colorable, and was one of the initial problems R.M.Karp [7] showed to be NP-complete. For fixed $k \geq 1$, the k -COLORING problem is deciding whether a given graph is k -colorable. Since Stockmeyer [13] showed that for any $k \geq 3$ the k -COLORING problem is NP-complete, there has been much interest in deciding for which classes of graphs coloring problems can be solved in polynomial time. In this paper, the general approach that we consider is to fix a graph H and consider the k -COLORING problem restricted to the class of H -free graphs.

We call a graph *acyclic* if it is C_n -free for all $n \geq 3$. The *girth* of a graph is the length of its shortest cycle, or infinity if the graph is acyclic. Kamiński and Lozin [9] proved:

1.1. *For any fixed $k, g \geq 3$, the k -COLORING problem is NP-complete for the class of graphs with girth at least g .*

As a consequence of 1.1, it follows that if the graph H contains a cycle, then for any fixed $k \geq 3$, the k -COLORING problem is NP-complete for the class of H -free graphs. The *claw* is the graph with vertex set $\{a_0, a_1, a_2, a_3\}$ and edge set $\{a_0a_1, a_0a_2, a_0a_3\}$. A theorem of Holyer [4] together with an extension due to Leven and Galil [10] imply the following:

1.2. *If a graph H contains the claw, then for every $k \geq 3$, the k -COLORING problem is NP-complete for the class of H -free graphs.*

Hence, the remaining problem of interest is deciding the k -COLORING problem for the class of H -free graphs where H is a fixed acyclic claw-free graph. It is easily observed

that every component of an acyclic claw-free graph is a path. And so, we focus on the k -COLORING problem for the class of H -free graphs where H is a connected acyclic claw-free graph, that is, simply a path. Hoàng, Kamiński, Lozin, Sawada, and Shu [5] proved the following:

1.3. *For every k , the k -COLORING problem can be solved in polynomial time for the class of P_5 -free graphs.*

Additionally, Randerath and Schiermeyer [11] showed that:

1.4. *The 3-COLORING problem can be solved in polynomial time for the class of P_6 -free graphs.*

While, Huang [6] recently showed that:

1.5. *The following problems are NP-complete:*

1. *The 5-COLORING problem is NP-complete for the class of P_6 -free graphs.*
2. *The 4-COLORING problem is NP-complete for the class of P_7 -free graphs.*

Thus, the remaining open cases of the k -COLORING problem for P_ℓ -free graphs are the following:

1. The 4-COLORING problem for the class of P_6 -free graphs.
2. The 3-COLORING problem for the class of P_ℓ -free graphs where $\ell \geq 7$.

Toward extending these polynomial results, it is convenient to consider the following more general coloring problem. A *palette* L of a graph G is a mapping which assigns each vertex $v \in V(G)$ a finite non-empty subset of \mathbb{N} , denoted by $L(v)$. A *subpalette* of a palette L of G is a palette L' of G such that $L'(v) \subseteq L(v)$ for all $v \in V(G)$. We say a palette L of the graph G has *order* k if $L(v) \subseteq \{1, \dots, k\}$ for all $v \in V(G)$. Notationally, we write (G, L) to represent a graph G and a palette L of G . We say that a k -coloring c of G is a *coloring* of (G, L) provided $c(v) \in L(v)$ for all $v \in V(G)$. We say (G, L) is *colorable*, if there exists a coloring of (G, L) . We denote by (G, \mathcal{L}) a graph G and a collection \mathcal{L} of palettes of G . We say (G, \mathcal{L}) is *colorable* if (G, L) is colorable for some $L \in \mathcal{L}$, and c is a *coloring* of (G, \mathcal{L}) if c is a coloring of (G, L) for some $L \in \mathcal{L}$.

Let G be a graph. A subset D of $V(G)$ is called a *dominating set*, if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D . Given (G, L) , consider a subset $X \subseteq V(G)$ such that $|L(x)| = 1$ for all $x \in X$. For a subset $Y \subseteq V(G) \setminus X$, we say that we *update the palettes of the vertices in Y with respect to X* , if for all $y \in Y$ we set

$$L(y) = L(y) \setminus \left(\bigcup_{u \in N(y) \cap X \text{ with } |L(u)|=1} L(u) \right).$$

Note that updating can be carried out in time $O(|V(G)|^2)$.

By reducing to an instance of 2-SAT, which Aspvall, Plass and Tarjan [1] showed can be solved in linear time, Edwards [3] proved the following:

1.6. *There is an algorithm with the following specifications:*

Input: *A palette L of a graph G such that $|L(v)| \leq 2$ for all $v \in V(G)$.*

Output: *A coloring of (G, L) , or a determination that none exists.*

Running time: $O(|V(G)|^2)$.

Let G be a graph. A subset S of $V(G)$ is called *monochromatic* with respect to a given coloring c of G if $c(u) = c(v)$ for all $u, v \in S$. For a palette L , and a set X of subsets of $V(G)$, we say that (G, L, X) is *colorable* if there is a coloring c of (G, L) such that S is monochromatic with respect to c for all $S \in X$. The proof of 1.6 is easily modified to obtain the following generalization [12]:

1.7. *There is an algorithm with the following specifications:*

Input: *A palette L of a graph G such that $|L(v)| \leq 2$ for all $v \in V(G)$, together with a set X of subsets of $V(G)$.*

Output: *A coloring of (G, L, X) , or a determination that none exists.*

Running time: $O(|X||V(G)|^2)$.

Applying 1.6 yields the following general approach for 3-coloring a graph. Let G be a graph, and suppose $D \subseteq V(G)$ is a dominating set. Initialize the order 3 palette L of G by setting $L(v) = \{1, 2, 3\}$ for all $v \in V(G)$. Consider a fixed 3-coloring c of $G[D]$, and let L_c be the subpalette of L obtained by updating the palettes of the vertices in $V(G) \setminus D$ with respect to D . By construction, (G, L_c) is colorable if and only if the coloring c of $G[D]$ can be extended to a 3-coloring of G . Since $|L_c(v)| \leq 2$ for all $v \in V(G)$, 1.6 allows us to efficiently test if (G, L_c) is colorable. Let \mathcal{L} to be the set of all such palettes L_c where c is a 3-coloring of $G[D]$. It follows that G is 3-colorable if and only if (G, \mathcal{L}) is colorable. Assuming we can efficiently produce a dominating set D of bounded size, since there are at most $3^{|D|}$ ways to 3-color $G[D]$, it follows that we can efficiently test if (G, \mathcal{L}) is colorable, and so we can decide if G is 3-colorable in polynomial time. This method figures prominently in the polynomial time algorithms for the 3-COLORING problem for the class of P_ℓ -free graphs where $\ell \leq 5$. However, this approach needs to be modified when considering the class of P_ℓ -free graphs when $\ell \geq 6$, since a dominating set of bounded size may not exist. Very roughly, the techniques used in this paper may be described as such a modification.

In this paper and [2], we prove that the 3-COLORING problem can be solved in polynomial time for the class of P_7 -free graphs. Here we consider the triangle-free case and prove the following:

1.8. *There is an algorithm with the following specifications:*

Input: *A $\{P_7, C_3\}$ -free graph G .*

Output: *A 3-coloring of G , or a determination that none exists.*

Running time: $O(|V(G)|^{18})$.

Here is a brief outline of the algorithm. Consider a $\{P_7, C_3\}$ -free graph G . We begin by establishing two polynomial time procedures 2.3 and 3.4 which determine if a 3-coloring of a specific induced subgraph of G extends to a coloring of G , and gives an explicit 3-coloring if one exists. More specifically, given an order 3 palette L of G , and a set X of subsets of $V(G)$, 2.3 and 3.4 allow us to reduce determining if (G, L, X) is colorable to determining if one of polynomially many triples (G', L', X') is colorable, where each of (G, L', X) is “closer” than (G, L, X) to being of the form required by 1.7. Next, we introduce a polynomial time “cleaning” procedure 4.3, which preprocesses the graph G so that we can apply 2.3 and 3.4. Next, we use 3.4 to show that if G contains a 7-gon, then in polynomial time we can either produce a 3-coloring of G , or determine that none exists. And so, we may assume G is a $\{P_7, C_3, C_7\}$ -free graph. Next, we use 3.4 to show that if G contains a shell, then in polynomial time we can either produce a 3-coloring of G , or determine that none exists. And so, we may assume G is a $\{P_7, C_3, C_7, shell\}$ -free graph. Finally, we use 2.3 to show that if G contains a 5-gon, then in polynomial time we can either produce a 3-coloring of G , or determine that none exists. And so, we may assume G is a $\{P_7, C_3, C_5, C_7\}$ -free graph. Since G is P_7 -free, it follows that G is C_k -free for all $k > 7$. And so, G is bipartite, and we can easily produce a 2-coloring of G , thus, establishing 1.8.

In [2], using different techniques, we prove the following:

1.9. *There is a polynomial time algorithm with the following specifications:*

Input: *A P_7 -free graph G which contains a triangle.*

Output: *A 3-coloring of G , or a determination that none exists.*

Together, 1.8 and 1.9, imply the following:

1.10. *There is a polynomial time algorithm with the following specifications:*

Input: *A P_7 -free graph G .*

Output: *A 3-coloring of G , or a determination that none exists.*

This paper is organized as follows. In section 2 we prove 2.3 and in section 3 we prove 3.4. In section 4, we give a preprocessing procedure 4.3, so that we can apply 2.3 and 3.4 to a given $\{P_7, C_3\}$ -free graph. In section 5 we prove a lemma that allows us to identify more easily situations where 2.3 and 3.4 are applicable. In section 6, we prove 6.6, which shows that if a $\{P_7, C_3\}$ -free graph contains a 7-gon, then 3-COLORING can be solved in polynomial time. In section 7, we prove 7.7, which shows that if a $\{P_7, C_3, C_7\}$ -free graph contains a shell, then 3-COLORING can be solved in polynomial time. In section 8, we prove 8.6, which shows that if a $\{P_7, C_3, C_7, \text{shell}\}$ -free graph contains a 5-gon, then 3-COLORING can be solved in polynomial time. Finally, in section 9, we tie everything together and give a formal proof of 1.8.

2 Reducing the Palettes: Part I

In this section, we give a polynomial time procedure 2.3 which, given a $\{P_7, C_3\}$ -free graph G with palette L , and a set of subsets X of $V(G)$, under certain circumstances, allows us to reduce determining if (G, L, X) is colorable to determining if one of polynomially many triples (G, L', X) is colorable, where each (G, L', X) is “closer” than (G, L, X) to the form required by 1.7. More precisely, more vertices have lists of size at most two in the palette L' than in the palette L . We begin with some definitions, and easy lemma and algorithm.

Let G be a graph. A *clique* in G is a set of vertices all pairwise adjacent. A *stable set* in G is a set of vertices all pairwise non-adjacent. The *neighborhood* of a vertex $v \in V(G)$ is the set of all vertices adjacent to v , and is denoted $N(v)$. The *degree* of a vertex $v \in V(G)$ is $|N(v)|$, and is denoted $\deg(v)$. A *partition* of a set S is a collection of disjoint subsets of S whose union is S . Let A and B be disjoint subsets of $V(G)$. For a vertex $b \in V(G) \setminus A$, we say that b is *complete* to A if b is adjacent to every vertex of A , and that b is *anticomplete* to A if b is non-adjacent to every vertex of A . If every vertex of A is complete to B , we say A is *complete* to B , and if every vertex of A is anticomplete to B , we say that A is *anticomplete* to B . If $b \in V(G) \setminus A$ is neither complete nor anticomplete to A , we say that b is *mixed* on A . We say G is *connected* if $V(G)$ cannot be partitioned into two disjoint non-empty sets anticomplete to each other. The *complement* \overline{G} of G is the graph with vertex set $V(G)$ such that two vertices are adjacent in \overline{G} if and only if they are non-adjacent in G . If \overline{G} is connected we say that G is *anticonnected*. For $X \subseteq V(G)$, we say that X is *connected* if $G[X]$ is connected, and that X is *anticonnected* if $G[X]$ is anticonnected. A *component* of $X \subseteq V(G)$ is a maximal connected subset of X , and an *anticomponent* of X is a maximal anticonnected subset of X .

2.1. *Let G be a bipartite $\overline{C_4}$ -free graph with bipartition (A, B) . If $a, a' \in A$ are such that $\deg(a) \leq \deg(a')$, then $N(a) \subseteq N(a')$.*

Proof. Suppose not, and so there exists $b \in N(a) \setminus N(a')$. Since $|N(a)| \leq |N(a')|$, it follows that there exists $b' \in N(a') \setminus N(a)$. However, then $\{a, b, a', b'\}$ is a $\overline{C_4}$ in G , a

contradiction. This proves 2.1. □

2.2. *There is an algorithm with the following specifications:*

Input: *A bipartite $\overline{C_4}$ -free graph G together with a bipartition $V(G) = A \cup B$.*

Output: *A partition $A_1 \cup \dots \cup A_q$ of A and an ordering $\{b_1, \dots, b_{|B|}\}$ of the vertices in B such that for every $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, |B|\}$ the following hold:*

1. *If $a, a' \in A_i$, then $N(a) = N(a')$, and*
2. *If b_j is complete to A_i , then $A_i \cup \dots \cup A_q$ is complete to $\{b_j, \dots, b_{|B|}\}$.*

Running time: $O(|V(G)|^2)$.

Proof. In time $O(|V(G)|^2)$ we can compute the degree of each vertex in G , and sort the vertices of A and B by degree, thus obtaining a labeling $a_1, \dots, a_{|A|}$ of A such that $\deg(a_1) \leq \dots \leq \deg(a_{|A|})$, and a labeling $b_1, \dots, b_{|B|}$ of B such that $\deg(b_1) \leq \dots \leq \deg(b_{|B|})$. Now, let $q = \deg(a_{|A|})$, and for each $i \in \{1, \dots, q\}$ define $A_i = \{a \in A : \deg(a) = i\}$. By applying 2.1 twice, it follows that if $a, a' \in A_i$, then $N(a) = N(a')$. Next, suppose b_j is complete to A_i for some $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, |B|\}$, which implies $A_i \subseteq N(b_j)$ and $b_j \in N(a)$ for all $a \in A_i$. Since $\deg(b_j) \leq \dots \leq \deg(b_{|B|})$, by 2.1, it follows that A_i is complete to $\{b_j, \dots, b_{|B|}\}$. And, since $\deg(a) \geq i$ for all $a \in A_i \cup \dots \cup A_q$, by 2.1, it follows that $\{b_j, \dots, b_{|B|}\}$ is complete to $A_i \cup \dots \cup A_q$. This proves 2.2. □

The following is the main result of the section.

2.3. *Let G be a $\{P_7, C_3\}$ -free graph with $V(G) = \{x\} \cup S \cup \hat{A} \cup \hat{B} \cup Y$, where*

- *x is complete to S and anticomplete to $\hat{A} \cup \hat{B} \cup Y$,*
- *$\hat{A} = \{a_1, \dots, a_t\}$ and $\hat{B} = \{b_1, \dots, b_t\}$ are stable,*
- *for $i, j \in \{1, \dots, t\}$, a_i is adjacent to b_j if and only if $i = j$, and*
- *each vertex of $\hat{A} \cup \hat{B}$ has a neighbor in S .*

Let L be an order 3 palette of G such that $L(v) \subseteq \{2, 3\}$ for every $v \in S$.

Let X be a set of subsets of $V(G)$.

Then there exists a set \mathcal{L} of $O(|V(G)|^2)$ subpalettes of L such that

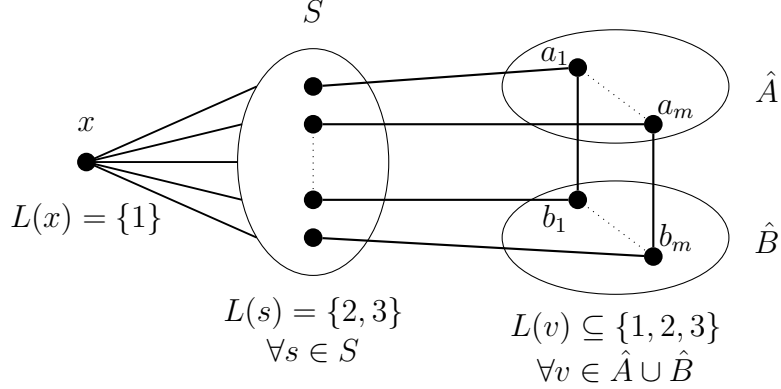


Figure 2: By 2.3, when we encounter the above situation we can reduce determining if (G, L, X) is colorable to determining if one of the triples (G, L', X) is colorable for some $L' \in \mathcal{L}$, where each of (G, L', X) is “closer” to the form required by 1.7 (in particular, $L'(v) \leq 2$ for all $v \in \hat{A} \cup \hat{B}$).

(a) For each $L' \in \mathcal{L}$, $L'(v) = L(v)$ for every $v \in \{x\} \cup S \cup Y$, and $|L'(v)| \leq 2$ for every $v \in \hat{A} \cup \hat{B}$, and

(b) (G, L, X) is colorable if and only if (G, L', X) is colorable for at least one $L' \in \mathcal{L}$; and for every $L' \in \mathcal{L}$, every coloring of (G, L', X) is a coloring of (G, L, X) .

Moreover, if the partition $\{x\} \cup S \cup \hat{A} \cup \hat{B} \cup Y$ of $V(G)$ is given, then \mathcal{L} can be computed in time $O(|V(G)|^4)$.

Proof. Since G is triangle-free, it follows that S is stable, and that every vertex of S is either anticomplete to or mixed on $\{a_i, b_i\}$ for every $i \in \{1, \dots, t\}$. Let H be a bipartite graph with bipartition $V(H) = S \cup \{c_1, \dots, c_t\}$, where $s \in S$ is adjacent to c_i in H if and only if s is mixed on $\{a_i, b_i\}$ in G . Note, H can be constructed in time $O(|V(G)|^2)$.

(1) H is a $\overline{C_4}$ -free graph.

Proof: Suppose not. Then there exist $s, s' \in S$ such that in G for $i \neq j$, s is mixed on $\{a_i, b_i\}$ and anticomplete to $\{a_j, b_j\}$, and s' is mixed on $\{a_j, b_j\}$ and anticomplete to $\{a_i, b_i\}$. By symmetry, we may assume that s is adjacent to a_i , and s' is adjacent to a_j . However, then $b_i - a_i - s - x - s' - a_j - b_j$ is a P_7 in G , a contradiction. This proves (1).

Write $C = \{c_1, \dots, c_t\}$. By (1), applying 2.2 in time $O(|V(G)|^2)$ we obtain a partition $S_1 \cup \dots \cup S_q$ of S and an ordering $\{d_1, \dots, d_t\}$ of the vertices of C . Renumber the vertices of \hat{A} and \hat{B} so that d_k corresponds to the edge $a_k b_k$ for every $k \in \{1, \dots, t\}$.

(2) For every $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, t\}$ the following hold:

(2a) The vertices in S_i are either all anticomplete to or all mixed on $\{a_j, b_j\}$.

(2b) If the vertices in S_i are all mixed on $\{a_j, b_j\}$, then every vertex in $S_i \cup \dots \cup S_q$ is mixed on $\{a_k, b_k\}$ for all $k \in \{j, \dots, t\}$.

Proof: By 2.2.1, it follows that in H every vertex d_j is either complete or anticomplete to S_i . Hence, by the construction of H , in G the vertices in S_i are either all anticomplete to or all mixed on $\{a_j, b_j\}$. This proves (2a). By 2.2.2, it follows that in H if d_j is complete to S_i , then $\{d_j, \dots, d_t\}$ is complete to $S_i \cup \dots \cup S_q$, and (2b) follows. This proves (2).

For $j \in \{1, \dots, t\}$, we define the *height* of the edge $a_j b_j$ to be the maximum ℓ such that both a_j and b_j have neighbors in $S_\ell \cup \dots \cup S_q$. Since every vertex in $\hat{A} \cup \hat{B}$ has a neighbor in S , the height of an edge is well defined. If the height of the edge $\{a_j, b_j\}$ is $\ell < q$, then (2) implies that one of the vertices in $\{a_j, b_j\}$ is anticomplete to $S_{\ell+1} \cup \dots \cup S_q$, we call this the *small* vertex in $\{a_j, b_j\}$ and denote it by s_j . We call the vertex of $\{a_j, b_j\} \setminus \{s_j\}$ the *large* vertex in $\{a_j, b_j\}$ and denote it by l_j . Then l_j is complete to $S_{\ell+1} \cup \dots \cup S_q$. If the edge $a_j b_j$ has height q , then we arbitrarily assign $\{l_j, s_j\} = \{a_j, b_j\}$. Next, let N_j be the set of vertices in $S_\ell \cup \dots \cup S_q$ adjacent to l_j , and let M_j be the set of vertices in $S_\ell \cup \dots \cup S_q$ adjacent to s_j . Clearly, computing the height of $a_j b_j$, determining the small and large vertices, and computing the N_j and M_j can be done in time $O(|V(G)|^2)$.

(3) For $j \in \{1, \dots, t\}$, suppose the edge $a_j b_j$ has height ℓ . Then the following hold:

(3a) $N_j \cup M_j = S_\ell \cup \dots \cup S_q$, where N_j, M_j are disjoint, both non-empty, and $M_j \subseteq S_\ell$.

(3b) Let $k \in \{1, \dots, t\} \setminus \{j\}$, and let $\{y, z\} = \{a_k, b_k\}$. If y is anticomplete to $S_\ell \cup \dots \cup S_q$, then the height of $a_k b_k$ is strictly less than ℓ , $y = s_k$, and both N_j, M_j are proper subsets of N_k .

Proof: Since G is triangle-free, it follows that N_j, M_j are disjoint. By the definition of height, both N_j, M_j are non-empty and, by (2a), it follows that every vertex in S_ℓ is mixed on $\{a_j, b_j\}$. Hence, by (2b), it follows that every vertex in $S_\ell \cup \dots \cup S_q$ is mixed on $\{a_j, b_j\}$, and so $N_j \cup M_j = S_\ell \cup \dots \cup S_q$. Finally, by our choice of s_j , it follows that $M_j \subseteq S_\ell$. This proves (3a). Next, we prove (3b). Since y is anticomplete to $S_\ell \cup \dots \cup S_q$, it follows, by the definition of height, that the height of $a_k b_k$ is strictly less than ℓ , and that $y = s_k$. Hence, by (3a), it follows that l_k is complete to $S_\ell \cup \dots \cup S_q$, and so both N_j, M_j are proper subsets of N_k . This proves (3b).

We say that (G, L, X) has a *type I coloring* if there exists a coloring c of (G, L, X) such that $\{c(a_i), c(b_i)\} = \{2, 3\}$ for some $i \in \{1, \dots, t\}$. We now prove the following:

(4) There exists a set \mathcal{L}_1 of $O(|V(G)|)$ of subpalettes of L such that

(4a) For each $L_1 \in \mathcal{L}_1$, $L_1(v) = L(v)$ for every $v \in \{x\} \cup S \cup Y$, and $|L_1(v)| \leq 2$ for every $v \in \hat{A} \cup \hat{B}$, and

(4b) (G, L, X) has a type I coloring if and only if (G, L_1, X) is colorable for some $L_1 \in \mathcal{L}_1$; and for every $L_1 \in \mathcal{L}_1$, every coloring of (G, L_1, X) is a type I coloring of (G, L, X) .

Moreover, \mathcal{L}_1 can be constructed in time $O(|V(G)|^3)$.

Proof: Let $i \in \{1, \dots, t\}$ and ℓ be the height of the edge $a_i b_i$. First, set

- $L_i(l_i) = \{2\}$,
- $L_i(s_i) = \{3\}$, and
- $L_i(v) = L(v)$ for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$.

Next, for each $j \in \{1, \dots, t\} \setminus \{i\}$, let $y \in \{a_j, b_j\}$. If $N(y) \cap (S_\ell \cup \dots \cup S_q) \neq \emptyset$, then set

$$L_i(y) = \begin{cases} L(y) \setminus \{3\} & , \text{ if } N(y) \cap (S_\ell \cup \dots \cup S_q) \subseteq N_i \\ L(y) \setminus \{2\} & , \text{ if } N(y) \cap (S_\ell \cup \dots \cup S_q) \subseteq M_i \\ \{1\} & , \text{ otherwise} \end{cases}$$

Otherwise, if y is anticomplete to $S_\ell \cup \dots \cup S_q$, then set $L_i(y) = \{2, 3\}$. As above, construct the subpalette L'_i of L , but with the roles of the colors 2 and 3 exchanged.

Clearly, if one of (G, L_i, X) and (G, L'_i, X) is colorable, then there exists a type I coloring of (G, L, X) . Now, suppose c is a type I coloring of (G, L, X) with $\{c(a_i), c(b_i)\} = \{2, 3\}$. By symmetry, we may assume $c(l_i) = 2$ and $c(s_i) = 3$. We claim that $c(v) \in L_i(v)$ for all $v \in V(G)$. By definition, this is the case for a_i, b_i , and for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$. Since $L_i(v) = L(v) \subseteq \{2, 3\}$ for every $v \in S$, as $c(l_i) = 2$, it follows that every vertex in N_i is colored 3. Similarly, as $c(s_i) = 3$, it follows that every vertex in M_i is colored 2. Hence, by (3a), the colors of all the vertices in $S_\ell \cup \dots \cup S_q$ are forced by $c(l_i)$ and $c(s_i)$. Next, consider $j \in \{1, \dots, t\} \setminus \{i\}$, and let $\{y, z\} = \{a_j, b_j\}$. If y has a neighbor $y' \in S_\ell \cup \dots \cup S_q$, then, as $c(y) \neq c(y')$, it follows that $c(y) \in L_i(y)$ by construction. So we may assume that y is anticomplete to $S_\ell \cup \dots \cup S_q$, and so, by construction, $L_i(y) = \{2, 3\}$. By (3b), it follows that the height of $a_j b_j$ is strictly less than ℓ , $y = s_j$, and that both N_i, M_i are proper subsets of N_j . Hence, by construction, $L_i(l_j) = \{1\}$. By (3a), both N_i, M_i are non-empty, and so $c(l_j) = 1$, which implies $c(s_j) \in \{2, 3\}$, and the claim holds.

For every $i \in \{1, \dots, t\}$, construct the subpalettes L_i, L'_i of L as above. Then $\mathcal{L}_1 = \{L_1, \dots, L_t, L'_1, \dots, L'_t\}$ satisfies (4a) and (4b). For a fixed $i \in \{1, \dots, t\}$, the subpalettes L_i, L'_i of L can be constructed in time $O(|V(G)|^2)$, and so \mathcal{L}_1 can be constructed in time $O(|V(G)|^3)$. This proves (4).

We say that (G, L, X) has a *type II coloring* if there exist distinct $i, j \in \{1, \dots, t\}$, $z \in \{a_j, b_j\}$ and a coloring c of (G, L, X) such that

(II.1) $a_i b_i$ and $a_j b_j$ have the same height ℓ , and

(II.2) N_i is not a proper subset of $N(z) \cap (S_\ell \cup \dots \cup S_q)$, and

(II.3) writing $\{y\} = \{a_j, b_j\} \setminus \{z\}$, we have $c(s_i) = c(z) = 1$, and $\{c(l_i), c(y)\} = \{2, 3\}$.

We now prove the following:

(5) There exists a set \mathcal{L}_2 of $O(|V(G)|^2)$ subpalettes of L such that

(5a) For each $L_2 \in \mathcal{L}_2$, $L_2(v) = L(v)$ for every $v \in \{x\} \cup S \cup Y$, and $|L_2(v)| \leq 2$ for every $v \in \hat{A} \cup \hat{B}$, and

(5b) (G, L, X) has a type II coloring if and only if (G, L_2, X) is colorable for some $L_2 \in \mathcal{L}_2$; and for every $L_2 \in \mathcal{L}_2$, every coloring of (G, L_2, X) is a type II coloring of (G, L, X) .

Moreover, \mathcal{L}_2 can be constructed in time $O(|V(G)|^4)$.

Proof: Let $i, j \in \{1, \dots, t\}$ be distinct, $\{y, z\} = \{a_j, b_j\}$, and suppose (II.1) and (II.2) are satisfied. First, set

- $L_{\{i,j\}}(l_i) = \{2\}$,
- $L_{\{i,j\}}(y) = \{3\}$,
- $L_{\{i,j\}}(s_i) = L_{\{i,j\}}(z) = \{1\}$, and
- $L_{\{i,j\}}(v) = L(v)$ for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$.

Next, for every $k \in \{1, \dots, t\} \setminus \{i, j\}$, consider each $w \in \{a_k, b_k\}$. If $N(w) \cap (S_\ell \cup \dots \cup S_q) \neq \emptyset$, then set

$$L_{\{i,j\}}(w) = \begin{cases} L(w) \setminus \{3\} & , \quad \text{if } N(w) \cap (S_\ell \cup \dots \cup S_q) \subseteq N_i \\ L(w) \setminus \{2\} & , \quad \text{if } N(w) \cap (S_\ell \cup \dots \cup S_q) \subseteq M_i \\ \{1\} & , \quad \text{otherwise} \end{cases}$$

Otherwise, if w is anticomplete to $S_\ell \cup \dots \cup S_q$, then set $L_{\{i,j\}}(w) = \{2, 3\}$. As above, construct the subpalette $L'_{\{i,j\}}$ of L , but with the roles of the colors 2 and 3 exchanged.

Clearly, if one of $(G, L_{\{i,j\}}, X)$ and $(G, L'_{\{i,j\}}, X)$ is colorable, then there exists a type II coloring of (G, L, X) . Now, suppose c is a type II coloring of (G, L, X) with $c(s_i) = c(z) = 1$, and $\{c(l_i), c(y)\} = \{2, 3\}$. By symmetry, we may assume $c(l_i) = 2$ and $c(y) = 3$. We claim that $c(v) \in L_{\{i,j\}}(v)$ for all $v \in V(G)$. By definition, this is the case for a_i, b_i, a_j, b_j , and for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$. Since $L_{\{i,j\}}(v) = L(v) \subseteq \{2, 3\}$ for every $v \in S$, as $c(l_i) = 2$, it follows, that every vertex in N_i is colored 3. Let $M' = N(y) \cap (S_\ell \cup \dots \cup S_q)$ and $N' = N(z) \cap (S_\ell \cup \dots \cup S_q)$. Similarly, as $c(y) = 3$, it follows that every vertex in M' is colored 2. Hence, $N_i \cap M' = \emptyset$. By (3a), $N_i \cup M_i = N' \cup M'$. Since N_i is not a

proper subset of N' , it follows that $N_i = N'$ and $M_i = M'$. And so, it follows that the colors of all the vertices in $S_\ell \cup \dots \cup S_q$ are forced; namely $c(v) = 3$ for every $v \in N_i$, and $c(v) = 2$ for every $v \in M_i$. Next, consider $k \in \{1, \dots, t\} \setminus \{i, j\}$, and let $u \in \{a_k, b_k\}$. If u has a neighbor $u' \in S_\ell \cup \dots \cup S_q$, then, as $c(u) \neq c(u')$, it follows that $c(u) \in L_{\{i,j\}}(u)$ by construction. So we may assume that u is anticomplete to $S_\ell \cup \dots \cup S_q$, and so, by construction, $L_{\{i,j\}}(u) = \{2, 3\}$. By (3b), it follows that the height of $a_k b_k$ is strictly less than ℓ , $u = s_k$, and that both N_i, M_i are proper subsets of N_k . Hence, by construction, $L_i(l_k) = \{1\}$. By (3a), both N_i, M_i are non-empty, and so $c(l_k) = 1$, which implies $c(s_k) \in \{2, 3\}$, and the claim holds.

For every distinct pair $i, j \in \{1, \dots, t\}$ satisfying (II.1) and (II.2), construct the subpalettes $L_{\{i,j\}}, L'_{\{i,j\}}$ of L as above. Let \mathcal{L}_2 be the set of all the subpalettes thus constructed, and observe that $|\mathcal{L}_2| \leq 4n^2$. Then \mathcal{L}_2 satisfies (5a) and (5b). For distinct $i, j \in \{1, \dots, t\}$, the corresponding subpalettes can be constructed in time $O(|V(G)|^2)$, and so \mathcal{L}_2 can be constructed in time $O(|V(G)|^4)$. This proves (5).

We say that (G, L, X) has a *type III coloring* if for some $i \in \{1, \dots, t\}$ where $a_i b_i$ has height ℓ , there exists a coloring c of (G, L, X) such that

(III.1) $c(l_i) \in \{2, 3\}$ and $c(s_i) = 1$,

(III.2) let $j \in \{1, \dots, t\} \setminus \{i\}$ such that the height of $a_j b_j$ is at most ℓ ; write $\{y, z\} = \{a_j, b_j\}$. If N_i is a proper subset of $N(z) \cap (S_\ell \cup \dots \cup S_q)$, then $c(z) = 1$.

We now prove the following:

(6) Suppose (G, L, X) has no type I coloring, and no type II coloring. Then there exists a set \mathcal{L}_3 of $O(|V(G)|)$ subpalettes of L such that

(6a) For each $L_3 \in \mathcal{L}_3$, $L_3(v) = L(v)$ for every $v \in \{x\} \cup S \cup Y$, and $|L_3(v)| \leq 2$ for every $v \in \hat{A} \cup \hat{B}$, and

(6b) (G, L, X) has a type III coloring if and only if (G, L_3, X) is colorable for some $L_3 \in \mathcal{L}_3$; and for every $L_3 \in \mathcal{L}_3$, every coloring of (G, L_3, X) is a type III coloring of (G, L, X) .

Moreover, \mathcal{L}_3 can be constructed in time $O(|V(G)|^3)$.

Proof: Let $i \in \{1, \dots, t\}$ and ℓ be the height of the edge $a_i b_i$. First, set

- $L_i(l_i) = \{2\}$,
- $L_i(s_i) = \{1\}$, and
- $L_i(v) = L(v)$ for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$.

Next, for every $j \in \{1, \dots, t\} \setminus \{i\}$, consider each $y \in \{a_j, b_j\}$. If $N(y) \cap N_i \neq \emptyset$, then set $L_i(y) = L(y) \setminus \{3\}$. Otherwise, if y is anticomplete to N_i , then, taking $z \in \{a_j, b_j\} \setminus \{y\}$, set

$$L_i(y) = \begin{cases} L(y) \setminus \{1\} & , \quad \text{if } N_i \text{ is a proper subset of } N(z) \cap (S_\ell \cup \dots \cup S_q) \\ L(y) \setminus \{3\} & , \quad \text{if } N_i \text{ is not a proper subset of } N(z) \cap (S_\ell \cup \dots \cup S_q) \end{cases}$$

As above, construct the subpalette L'_i of L , but with the roles of the colors 2 and 3 exchanged.

First, we argue that every coloring of (G, L_i, X) and (G, L'_i, X) is a type III coloring of (G, L, X) . Suppose that one of (G, L_i, X) and (G, L'_i, X) is colorable. By symmetry, we may assume that c is a coloring of (G, L_i, X) , and so $c(l_i) = 2$, $c(s_i) = 1$, and (III.1) holds. Now, we show (III.2) holds. Suppose for $j \in \{1, \dots, t\} \setminus \{i\}$ the height ℓ' of $a_j b_j$ is at most ℓ . Fix $\{y, z\} = \{a_j, b_j\}$. Since $c(l_i) = 2$, every vertex in N_i is colored 3. Suppose N_i is a proper subset of $N(z) \cap (S_\ell \cup \dots \cup S_q)$, and so $c(z) \neq 3$. By (3a), it follows that y is anticomplete to N_i , and so, by construction, $c(y) \neq 1$. Thus, since (G, L, X) does not have a type I coloring, it follows that $c(z) = 1$, and so (III.2) holds. Hence, c is a type III coloring of (G, L, X) .

Next, we argue that if (G, L, X) has a type III coloring, then (G, L_3, X) is colorable for some $L_3 \in \mathcal{L}_3$. Suppose that c is a type III coloring of (G, L, X) with $c(l_i) \in \{2, 3\}$ and $c(s_i) = 1$. By symmetry, we may assume $c(l_i) = 2$. We claim that $c(v) \in L_i(v)$ for all $v \in V(G)$. By definition, this is the case for a_i, b_i , and for all $v \in V(G) \setminus (\hat{A} \cup \hat{B})$. Since $L_i(v) = L(v) \subseteq \{2, 3\}$ for every $v \in S$, as $c(l_i) = 2$, it follows that every vertex in N_i is colored 3. Next, consider $j \in \{1, \dots, t\} \setminus \{i\}$, and let $\{y, z\} = \{a_j, b_j\}$. If y has a neighbor in N_i , then $c(y) \neq 3$, and it follows that $c(y) \in L_i(y)$ by construction. So we may assume that y is anticomplete to N_i . By (3a), it follows that the height of $a_j b_j$ is at most ℓ , and that z is complete to N_i . Hence, $c(z) \in \{1, 2\}$. If N_i is a proper subset of $N(z) \cap (S_\ell \cup \dots \cup S_q)$, then $c(z) = 1$, by (III.2), and so $c(y) \neq 1$. Thus, we may assume that N_i is not a proper subset of $N(z) \cap (S_\ell \cup \dots \cup S_q)$. Since z is complete to N_i , this implies that $N_i = N(z) \cap (S_\ell \cup \dots \cup S_q)$. Hence, by (3a), it follows that $M_i = N(y) \cap (S_\ell \cup \dots \cup S_q)$, and so $a_j b_j$ also has height ℓ . Thus, since (G, L, X) does not admit a type II coloring, it follows that $c(y) \neq 3$, and the claim holds.

For every $i \in \{1, \dots, t\}$, construct the subpalettes L_i, L'_i of L as above. Then $\mathcal{L}_3 = \{L_1, \dots, L_t, L'_1, \dots, L'_t\}$ satisfies (6a) and (6b). For a fixed $i \in \{1, \dots, t\}$, the subpalettes L_i, L'_i of L can be constructed in time $O(|V(G)|^2)$, and so \mathcal{L}_3 can be constructed in time $O(|V(G)|^3)$. This proves (6).

Finally, define the additional subpalette \hat{L} of L such that for $v \in V(G)$

$$\hat{L}(v) = \begin{cases} L(v) & , \quad \text{if } v \in V(G) \setminus (\hat{A} \cup \hat{B}) \\ \{1\} & , \quad \text{if } v = l_i \text{ for some } i \in \{1, \dots, t\} \\ \{2, 3\} & , \quad \text{if } v = s_i \text{ for some } i \in \{1, \dots, t\} \end{cases}$$

Define $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \{\hat{L}\}$. Note, \mathcal{L} has size $O(|V(G)|^2)$, as it is dominated by \mathcal{L}_2 , and can be constructed in time $O(|V(G)|^4)$, by (4), (5) and (6). By (4a), (5a), and (6a), it follows that \mathcal{L} satisfies (a). We now argue that \mathcal{L} satisfies (b). Since \mathcal{L} contains only subpalettes of L , it clearly follows that if (G, L', X) is colorable for some $L' \in \mathcal{L}$, then (G, L, X) is colorable; and for every $L' \in \mathcal{L}$, every coloring of (G, L', X) is a coloring of (G, L, X) . Now, suppose that c is a coloring of (G, L, X) . If c is a type I or II coloring of (G, L, X) , then, by (4b) and (5b), it follows that (G, L', X) is colorable for some $L' \in \mathcal{L}_1 \cup \mathcal{L}_2$. Hence, we may assume that (G, L, X) admits no coloring of type I or II. If $c(l_i) = 1$ for every $i \in \{1, \dots, t\}$, then $c(s_i) \in \{2, 3\}$ for every $i \in \{1, \dots, t\}$, and it follows that c is a coloring of (G, \hat{L}, X) , so we may assume not.

We claim that c is a type III coloring of (G, L, X) . Let $a_i b_i$ be an edge with minimum height ℓ_{\min} such that $c(l_i) \in \{2, 3\}$, and subject to that with N_i maximal. By symmetry, we may assume $c(l_i) = 2$. Since c is not a type I coloring of (G, L, X) , it follows that $c(s_i) = 1$, and thus (III.1) is satisfied. By our choice of ℓ_{\min} and i , for all $j \in \{1, \dots, t\} \setminus \{i\}$ if the height of $a_j b_j$ is at most ℓ_{\min} and N_i is a proper subset of $N(z) \cap (S_\ell \cup \dots \cup S_q)$ for some $z \in \{a_j, b_j\}$, then $c(z) = 1$, and so (III.2) is satisfied. This proves that c is a type III coloring of (G, L, X) . Now, by (6b), it follows that (G, L', X) is colorable for some $L' \in \mathcal{L}_3$. This proves 2.3. □

3 Reducing the Palettes: Part II

Let G be a graph, L a palette of G , and X a set of subsets of $V(G)$. Let \mathcal{P} be a collection of triples (G', L', X') , where G' is an induced subgraph of G , L' is a subpalette of $L|_{V(G')}$, and X' is a set of subsets of $V(G')$. We say that \mathcal{P} is *colorable* if at least one element of \mathcal{P} is colorable. We say that \mathcal{P} is a *restriction* of (G, L, X) if \mathcal{P} being colorable implies that (G, L, X) is colorable, and we can extend a coloring of a colorable element of \mathcal{P} to a coloring of (G, L, X) in time $O(|V(G)|)$.

First, we make the following easy observation:

3.1. *Let G be a graph, and let $v \in V(G)$. Let L be an order k palette of a graph G , and let X be a set of subsets of $V(G) \setminus \{v\}$. If $|L(v)| = k$ and in every coloring of $(G \setminus v, L, X)$ at most $k - 1$ colors are used to color $N(v)$, then (G, L, X) is colorable if and only if $(G \setminus v, L|_{V(G) \setminus \{v\}}, X)$ is colorable. Furthermore, we can extend a coloring of $(G \setminus v, L|_{V(G) \setminus \{v\}}, X)$ to a coloring of (G, L, X) in constant time.*

Next, we prove the following:

3.2. *Let G be a triangle-free graph, and let $u, v \in V(G)$ be adjacent. Let X be a set of subsets of $V(G) \setminus \{u, v\}$. Let L be an order 3 palette of G . Assume that:*

- $|L(v)| = 3$, and in every coloring of $(G \setminus \{u, v\}, L|_{V(G) \setminus \{u, v\}}, X)$, at most two colors are used to color $N(v) \setminus \{u\}$, and

- $|L(u)| = 2$ with $L(y) \subseteq \{1, 2, 3\} \setminus L(u)$ for all $y \in N(u) \setminus \{v\}$.

Then (G, L, X) is colorable if and only if $(G \setminus \{u, v\}, L|_{V(G) \setminus \{u, v\}}, X)$ is colorable. Furthermore, we can extend a coloring of $(G \setminus \{u, v\}, L|_{V(G) \setminus \{u, v\}}, X)$ to a coloring of (G, L, X) in constant time.

Proof. Clearly, if (G, L, X) is colorable, then $(G \setminus \{u, v\}, L|_{V(G) \setminus \{u, v\}}, X)$ is colorable. Now, suppose c is a coloring of $(G \setminus \{u, v\}, L|_{V(G) \setminus \{u, v\}}, X)$. We may assume that only colors 1 and 2 are used on $N(v) \setminus \{u\}$. Assigning $c(v) = 3$ and $c(u) \in L(u) \setminus \{3\}$, we obtain a coloring of (G, L, X) . This proves 3.2. \square

Let Z be a set of subsets of $V(G)$. Given a coloring c of (G, L, Z) and a subset $X \subseteq V(G)$ define the subpalette L_c^X of L as follows: for every $v \in V(G)$, set

$$L_c^X(v) = \begin{cases} c(v) & , \quad \text{if } v \in X \\ L(v) & , \quad \text{otherwise} \end{cases}.$$

For a subset $Y \subseteq V(G) \setminus X$, let $L_c^{X,Y}$ be the subpalette of L_c^X obtained by updating the palettes of the vertices in Y with respect to X .

3.3. *If c is a coloring of (G, L, Z) , then c is a coloring of $(G, L_c^{X,Y}, Z)$.*

Proof. Clearly, c is a coloring of $(G, L_c^{X,Y}, Z)$, since, by the definition of updating, $c(v) \in L_c^{X,Y}(v)$ for all $v \in V(G)$. This proves 3.3. \square

A vertex u in a graph G is *dominated* if there is $v \in V(G) \setminus \{u\}$ such that u is non-adjacent to v , and $N(u) \subseteq N(v)$. In this case we say that u is *dominated by* v . The following is the main result of this section:

3.4. *Let L be an order 3 palette of a $\{P_7, C_3\}$ -free graph G , such that $V(G) = \tilde{A} \cup \tilde{B} \cup \tilde{C}$, where*

- \tilde{B} is anticomplete to \tilde{C} ,
- every component of \tilde{C} has at most 2 vertices,
- every vertex of \tilde{C} has a neighbor in \tilde{A} ,
- every vertex of \tilde{A} has a neighbor in \tilde{C} ,
- $|L(a)| \leq 2$ for every $a \in \tilde{A}$,
- $|L(c)| = 3$ for every $c \in \tilde{C}$, and

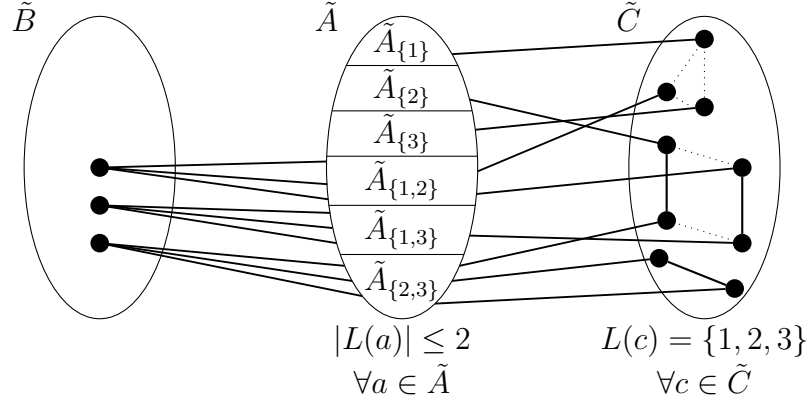


Figure 3: By 3.4, when we encounter the above situation we can reduce determining if (G, L, Z) is colorable to determining if a restriction \mathcal{P} of (G, L, Z) is colorable. The elements of \mathcal{P} are “closer” to being of the form required by 1.7 than (G, L, Z) is.

- G contains no dominated vertices.

For every non-empty subset $X \subseteq \{1, 2, 3\}$, let $\tilde{A}_X = \{a \in \tilde{A} \text{ with } L(a) = X\}$. For every $c \in \tilde{C}$ and distinct $i, j \in \{1, 2, 3\}$, let $N_{\{i,j\}}(c) = N(c) \cap \tilde{A}_{\{i,j\}}$, and $M_{\{i,j\}}(c) = \tilde{A}_{\{i,j\}} \setminus N(c)$.

Assume also that:

- For every $c_1, c_2 \in \tilde{C}$ and $\{i, j, k\} = \{1, 2, 3\}$, $N_{\{i,j\}}(c_1) \cap M_{\{i,j\}}(c_2)$ is complete to $M_{\{i,k\}}(c_1) \cap N_{\{i,k\}}(c_2)$.
- For distinct $i, j \in \{1, 2, 3\}$, there exists a vertex in \tilde{B} complete to $\tilde{A}_{\{i,j\}}$.

Let Z be a set of subsets of A . Then there exists a restriction \mathcal{P} of (G, L, Z) , of size $O(|V(G)|^7)$, such that

- (a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ for every $(G', L', X') \in \mathcal{P}$, and
- (b) Every $(G', L', X') \in \mathcal{P}$ is such that $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and $|X'|$ has size $O(|Z| + |V(G)|)$, and
- (c) If (G, L, Z) is colorable, then \mathcal{P} is colorable.

Moreover, if the partition $\tilde{A} \cup \tilde{B} \cup \tilde{C}$ of $V(G)$ is given, then \mathcal{P} can be computed in time $O(|V(G)|^7)$.

Proof. Since $|L(a)| \leq 2$ for all $a \in \tilde{A}$, we obtain the partition $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup \tilde{A}_{\{1,2\}} \cup \tilde{A}_{\{1,3\}} \cup \tilde{A}_{\{2,3\}}$ of \tilde{A} . Let $i, j \in \{1, 2, 3\}$ be distinct. Since G is triangle-free and there exists a vertex in \tilde{B} complete to $\tilde{A}_{\{i,j\}}$, it follows that $\tilde{A}_{\{i,j\}}$ is stable. Further, there is no vertex $c \in \tilde{C}$ such that $N(c) \subseteq \tilde{A}_{\{i,j\}}$, as such a vertex would be dominated by any vertex in \tilde{B} complete to $\tilde{A}_{\{i,j\}}$. Thus, it follows that:

(1) Every vertex $c \in \tilde{C}$ such that $N(c) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$ for some distinct $i, j \in \{1, 2, 3\}$ belongs to a component of \tilde{C} of size 2.

A coloring c of (G, L, Z) is a *type I coloring* if for distinct $i, j \in \{1, 2, 3\}$ there exists $z \in \tilde{C}$ and $x, y \in N_{\{i,j\}}(z)$ with $c(x) = i$ and $c(y) = j$.

(2) There exists a restriction \mathcal{P}_1 of (G, L, Z) , of size $O(|V(G)|^5)$, such that:

(2a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ and $X' = Z$ for every $(G', L', X') \in \mathcal{P}_1$,

(2b) Every $(G', L', X') \in \mathcal{P}_1$ is such that $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and

(2c) If (G, L, Z) has a type I coloring, then \mathcal{P}_1 is colorable.

Moreover, \mathcal{P}_1 can be computed in time $O(|V(G)|^7)$.

Proof: Let $z \in \tilde{C}$, $\{i, j, k\} = \{1, 2, 3\}$, and $x, y \in N_{\{i,j\}}(z)$. Let $U_{(x,y,z)}$ be the set of all vertices $u \in M_{\{i,k\}}(z) \cup M_{\{j,k\}}(z)$ for which there exists $c \in \tilde{C}$ such that c is adjacent to u and anticomplete to $\{x, y\}$. Since $x, y \in N_{\{i,j\}}(z)$, by assumption, it follows that $\{x, y\}$ is complete to $U_{(x,y,z)}$. Set

- $L'(v) = \{i\}$, for all $v \in \{x\} \cup N_{\{i,k\}}(z)$,
- $L'(v) = \{j\}$, for all $v \in \{y\} \cup N_{\{j,k\}}(z)$,
- $L'(v) = \{k\}$ for $v \in \{z\} \cup U_{(x,y,z)}$, and
- $L'(v) = L(v)$, for every $v \in V(G) \setminus (N_{\{i,k\}}(z) \cup N_{\{j,k\}}(z) \cup U_{(x,y,z)} \cup \{x, y, z\})$.

Let $A = \{x, y\} \cup (\tilde{A} \setminus \tilde{A}_{\{i,j\}})$. Next, update the palettes of the vertices in \tilde{C} with respect to A . If $c \in \tilde{C}$ has a neighbor in $(M_{\{i,k\}}(z) \cup M_{\{j,k\}}(z)) \setminus U_{(x,y,z)}$, then it follows, by the definition of $U_{(x,y,z)}$, that c is not anticomplete to $\{x, y\}$. And so after updating, for every $c \in \tilde{C}$ if $N(c) \cap (\tilde{A} \setminus \tilde{A}_{\{i,j\}})$ is non-empty, then $|L'(c)| \leq 2$. Let F be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,j\}}$. Initialize $D_{(x,y,z)}^{ij} = \emptyset$. Consider a vertex $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$. Then $|L'(c_1)| = 3$, and, by (1), c_1 is adjacent to some $c_2 \in \tilde{C}$ which is anticomplete to $\tilde{A}_{\{i,j\}}$.

Since every vertex of \tilde{C} has a neighbor in \tilde{A} , it follows that $|L'(c_2)| \leq 2$. If $|L'(c_2)| = 1$, set $L'(c_1) = L'(c_1) \setminus L'(c_2)$. Otherwise, $|L'(c_2)| = 2$. Since c_2 is anticomplete to $\tilde{A}_{\{i,j\}}$, it follows that that

$$N(c_2) \cap \tilde{A} \subseteq \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,k\}}(z) \cup N_{\{j,k\}}(z) \cup U_{(x,y,z)}.$$

In particular $|L'(v)| = 1$ for every $v \in N(c_2) \cap \tilde{A}$, and so, by the definition of updating, $L'(v) \subseteq \{1, 2, 3\} \setminus L(c_2)$ for all $v \in N(c_2) \setminus \{c_1\}$. Therefore, by 3.2, it follows that $(G \setminus D_{(x,y,z)}^{ij}, L'|_{V(G) \setminus D_{(x,y,z)}^{ij}}, Z)$ is colorable if and only if $(G \setminus (D_{(x,y,z)}^{ij} \cup \{c_1, c_2\}), L'|_{V(G) \setminus (D_{(x,y,z)}^{ij} \cup \{c_1, c_2\})}, Z)$ is colorable. In this case, set $D_{(x,y,z)}^{ij} = D_{(x,y,z)}^{ij} \cup \{c_1, c_2\}$. Carry out this procedure for every $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$. Let $G_{(x,y,z)}^{ij} = G \setminus D_{(x,y,z)}^{ij}$. Repeatedly applying the previous argument, it follows that (G, L', Z) is colorable if and only if $(G_{(x,y,z)}^{ij}, L'|_{V(G_{(x,y,z)}^{ij})}, Z)$ is colorable. By assumption, there exists $b \in \tilde{B}$ complete to $\tilde{A}_{\{i,j\}}$. Let \mathcal{L}' be the set of $O(|V(G)|^2)$ subpalettes of $L'|_{V(G_{(x,y,z)}^{ij})}$ obtained from 2.3 applied with

- $x = b$,
- $S = \tilde{A}_{\{i,j\}}$,
- $\hat{A} \cup \hat{B} = F$,
- $Y = V(G_{(x,y,z)}^{ij}) \setminus (\{b\} \cup \tilde{A}_{\{i,j\}} \cup F)$, and
- $X = Z$.

For each $\hat{L} \in \mathcal{L}'$, define the subpalette $L_{(x,y,z)}^{ij}(\hat{L})$ of $L|_{V(G_{(x,y,z)}^{ij})}$ as follows: For $v \in V(G_{(x,y,z)}^{ij})$ set

$$L_{(x,y,z)}^{ij}(\hat{L})(v) = \begin{cases} \hat{L}(v) & , \text{ if } v \in \tilde{C} \setminus D_{(x,y,z)}^{ij} \\ L'(v) & , \text{ otherwise} \end{cases}$$

Let

$$\mathcal{P}_{(x,y,z)}^{ij} = \{(G_{(x,y,z)}^{ij}, L_{(x,y,z)}^{ij}(\hat{L}), Z) : \hat{L} \in \mathcal{L}'\}.$$

Let \mathcal{P}_1 be the union of all $\mathcal{P}_{(x,y,z)}^{ij}$. Since there are at most $3|V(G)|^3$ choices of x, y, z and $\{i, j\}$, and each set F can be found in time $O(|V(G)|^2)$, it follows that building \mathcal{P}_1 requires $O(|V(G)|^3)$ applications of 2.3, and so \mathcal{P}_1 can be constructed in time $O(|V(G)|^7)$. Now, we argue that \mathcal{P}_1 is indeed a restriction (G, L, Z) . Suppose $\mathcal{P}_{(x,y,z)}^{ij}$ is colorable. By 2.3, it follows that $(G_{(x,y,z)}^{ij}, L'|_{V(G_{(x,y,z)}^{ij})}, Z)$ is colorable, and so, as observed above, by construction and 3.2, it follows that (G, L', Z) is colorable. Since L' is a subpalette of L , we deduce that (G, L, Z) is colorable. This proves that \mathcal{P}_1 is indeed a restriction (G, L, Z) .

By construction and 2.3, (2a) and (2b) hold. Next, we show that (2c) holds. We need to prove that if (G, L, Z) admits a type I coloring, then \mathcal{P}_1 is colorable. Suppose c is a type I coloring of (G, L, Z) . Let $z \in \tilde{C}$ with $c(z) = k$ and $x, y \in N_{\{i,j\}}(z)$ with $c(x) = i$ and $c(y) = j$. We claim that the restriction $\mathcal{P}_{(x,y,z)}^{ij}$ is colorable. Since z is complete to $N_{\{i,k\}}(z) \cup N_{\{j,k\}}(z)$, it follows that $c(v) = i$ for every $v \in N_{\{i,k\}}(z)$ and $c(v) = j$ for every $v \in N_{\{j,k\}}(z)$. By assumption, for every $c_1, c_2 \in \tilde{C}$ we have that $N_{\{i,j\}}(c_1) \cap M_{\{i,j\}}(c_2)$ is complete to $M_{\{i,k\}}(c_1) \cap N_{\{i,k\}}(c_2)$. Taking $c_1 = z$, it follows that $\{x, y\}$ is complete to every $v \in M_{\{i,k\}}(z) \cup M_{\{j,k\}}(z)$ for which there exists $c \in \tilde{C}$ such that c is adjacent to v and anticomplete to $\{x, y\}$, that is, $\{x, y\}$ is complete to $U_{(x,y,z)}$. Consequently, $c(v) = k$ for all $v \in U_{(x,y,z)}$. Also $c(v) = k$ for every $v \in \tilde{B}$ that is complete to $\tilde{A}_{\{i,j\}}$. By 3.3, it follows that c is a coloring of $(G, L_c^{A, \tilde{C}}, Z)$. Let F be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,j\}}$. Let C' be the set of vertices $v \in \tilde{C} \setminus F$ with $N(v) \cap \tilde{A} \subseteq \tilde{A}_{\{i,j\}}$. By construction, $c(v) \in L''(v)$ for all $L'' \in \mathcal{L}_{(x,y,z)}^{ij}$ and $v \in V(G_{(x,y,z)}^{ij}) \setminus (C' \cup F)$. Next, we show that the claim holds for every vertex in $C' \setminus D_{(x,y,z)}^{ij}$. By (1) and construction, every vertex $c_1 \in C' \setminus D_{(x,y,z)}^{ij}$ is adjacent to some $c_2 \in \tilde{C} \setminus D_{(x,y,z)}^{ij}$, and $|L_c^{A, \tilde{C}}(c_2)| = 1$. Since $c(c_1) \neq c(c_2)$, it follows that $c(c_1) \in L''(c_1)$ for all $L'' \in \mathcal{L}_{(x,y,z)}^{ij}$. Now $\mathcal{P}_{(x,y,z)}^{ij}$ is colorable by 2.3 and (2c) follows. This proves (2).

Let $\mathcal{X} = \{N(v) \cap \tilde{A}_{\{i,j\}} : v \in \tilde{C}, 1 \leq i < j \leq 3\}$, and let $\mathcal{Z} = Z \cup \mathcal{X}$. From (2) it follows that:

(3) If (G, L, Z) does not have a type I coloring, then (G, L, Z) is colorable if and only if (G, L, \mathcal{Z}) is colorable.

(4) Assume that (G, L, Z) does not have a type I coloring. If there exists a vertex in \tilde{C} with neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, then there exists a restriction \mathcal{P}_2 of size $O(|V(G)|^2)$ of (G, L, Z) such that:

(4a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ and $X' = \mathcal{Z}$ for every $(G', L', X') \in \mathcal{P}_2$,

(4b) Every $(G', L', X') \in \mathcal{P}_2$ is such that $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and

(4c) If (G, L, Z) is colorable, then \mathcal{P}_2 is colorable.

Moreover, \mathcal{P}_2 can be computed in time $O(|V(G)|^4)$.

If there does not exist a vertex in \tilde{C} with neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, then $\mathcal{P}_2 = \emptyset$.

Proof: Let $w \in \tilde{C}$. For $\{i, j, k\} = \{1, 2, 3\}$, let $P_{\{j,k\}}(w)$ be the set of vertices $v \in M_{\{j,k\}}(w)$ for which there exists $c \in \tilde{C}$ adjacent to v and anticomplete to $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$. By

assumption, it follows that $P_{\{j,k\}}(w)$ is complete to $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$. Thus, if $v \in M_{\{j,k\}}(w) \setminus P_{\{j,k\}}(w)$, then every vertex in $N(v) \cap \tilde{C}$ has a neighbor in $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$.

If there does not exist a vertex in \tilde{C} with neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, set $\mathcal{P}_2 = \emptyset$, and halt. Otherwise, suppose $w \in \tilde{C}$ has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$. Let $\{i, j, k\} = \{1, 2, 3\}$ be such that $P_{\{j,k\}}(w) = \emptyset$ (we will show later that if (G, L, Z) is colorable, then such $P_{\{j,k\}}(w)$ exists); if no such $\{i, j, k\}$ exists, set $\mathcal{P}_2 = \emptyset$, and halt. Set

- $L'(v) = \{i\}$, for all $v \in \{w\} \cup P_{\{i,j\}}(w)$,
- $L'(v) = \{j\}$, for all $v \in N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w)$,
- $L'(v) = \{k\}$, for all $v \in N_{\{i,k\}}(w)$, and
- $L'(v) = L(v)$, for every $v \in V(G) \setminus (N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w) \cup P_{\{i,j\}}(w) \cup \{w\})$.

Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w) \cup P_{\{i,j\}}(w)$. Next, update the palettes of all the vertices in \tilde{C} with respect to A . If $c \in \tilde{C}$ has a neighbor in $(M_{\{i,j\}}(w) \cup M_{\{j,k\}}(w)) \setminus P_{\{i,j\}}$, then c is not anticomplete to $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w)$, and so after updating, for every $c \in \tilde{C}$, if $N(c) \cap (\tilde{A} \setminus M_{\{i,k\}}(w)) \neq \emptyset$, then $|L'(v)| \leq 2$. Let F to be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $M_{\{i,k\}}(w)$. Initialize $D_{(i,j,k)} = \emptyset$. Consider a vertex $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w)$. Then $|L'(c_1)| = 3$, and, by (1), c_1 is adjacent to some $c_2 \in \tilde{C}$. Since in every coloring of $(G \setminus c_1, L', \mathcal{Z})$ at most two colors appear in $N(c_1) \subseteq \{c_2\} \cup M_{\{i,k\}}(w)$, by 3.1, it follows that $(G \setminus D_{(i,j,k)}, L'|_{V(G) \setminus D_{(i,j,k)}}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D_{(i,j,k)} \cup \{c_1\}), L'|_{V(G) \setminus (D_{(i,j,k)} \cup \{c_1\})}, \mathcal{Z})$ is colorable. In this case, set $D_{(i,j,k)} = D_{(i,j,k)} \cup \{c_1\}$. Carry out this procedure for every $c_1 \in \tilde{C} \setminus F$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w)$. Let $G_{(i,j,k)} = G \setminus D_{(i,j,k)}$. Repeatedly applying the previous argument, it follows that (G, L', \mathcal{Z}) is colorable if and only if $(G_{(i,j,k)}, L'|_{V(G_{(i,j,k)})}, \mathcal{Z})$ is colorable. By assumption, there exists $b \in \tilde{B}$ complete to $\tilde{A}_{\{i,k\}}$; therefore b is complete to $M_{\{i,k\}}(w)$. Let \mathcal{L}' be the set of $O(|V(G)|^2)$ subpalettes of $L'|_{V(G_{(i,j,k)})}$ obtained from 2.3 applied with

- $x = b$,
- $S = M_{\{i,k\}}(w)$,
- $\hat{A} \cup \hat{B} = F$,
- $Y = V(G_{(i,j,k)}) \setminus (\{b\} \cup M_{\{i,k\}}(w) \cup F)$, and
- $X = \mathcal{Z}$.

For each $\hat{L} \in \mathcal{L}'$, define the subpalette $L_{(i,j,k)}(\hat{L})$ of $L|_{V(G_{(i,j,k)})}$ as follows: For $v \in V(G_{(i,j,k)})$ set

$$L_{(i,j,k)}(\hat{L})(v) = \begin{cases} \hat{L}(v) & , \text{ if } v \in \tilde{C} \setminus D_{(i,j,k)} \\ L'(v) & , \text{ otherwise} \end{cases}$$

Let

$$\mathcal{P}_{(i,j,k)} = \{(G_{(i,j,k)}, L_{(i,j,k)}(\hat{L}), \mathcal{Z}) : \hat{L} \in \mathcal{L}'\}.$$

Let \mathcal{P}_2 be union of all $\mathcal{P}_{(i,j,k)}$. Since w can be found in time $O(|V(G)|^2)$, there are 6 choices of (i, j, k) , and each set F can be found in time $O(|V(G)|^2)$, it follows that building \mathcal{P}_2 requires 6 applications of 2.3, and so \mathcal{P}_2 can be constructed in time $O(|V(G)|^4)$. Now, we argue that \mathcal{P}_2 is indeed a restriction of (G, L, Z) . Suppose $\mathcal{P}_{(i,j,k)}$ is colorable. By 2.3, it follows that $(G_{(i,j,k)}, L'|_{V(G_{(i,j,k)})}, \mathcal{Z})$ is colorable, and so, as observed above, by construction and 3.1, it follows that (G, L', \mathcal{Z}) is colorable. Since L' is a subpalette of L , and $Z \subseteq \mathcal{Z}$, we deduce that (G, L, Z) is colorable.

By construction and 2.3, (4a) and (4b) hold. Next, we show that (4c) holds. Suppose c is a coloring of (G, L, Z) with $c(w) = i$ where $w \in \tilde{C}$ has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$. By (3), c is a coloring of (G, L, \mathcal{Z}) , and by symmetry, we may assume that $j, k \in \{1, 2, 3\} \setminus \{i\}$ are such that $c(v) = j$ for all $v \in N_{\{j,k\}}(w)$. We claim that the restriction $\mathcal{P}_{(i,j,k)}$ is colorable. Clearly, every set in \mathcal{Z} is monochromatic in c . It follows that $c(v) = j$ for every $v \in N_{\{i,j\}}(w)$, and $c(v) = k$ for every $v \in N_{\{i,k\}}(w)$. Since $N_{\{j,k\}}(w)$ is complete to $P_{\{i,j\}}(w)$, it follows that $c(v) = i$ for every $v \in P_{\{i,j\}}(w)$. Since $N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w)$ is complete to $P_{\{j,k\}}(w)$, it follows that $P_{\{j,k\}}(w)$ is empty. Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \cup N_{\{i,k\}}(w) \cup P_{\{i,j\}}(w)$.

By 3.3, it follows that c is a coloring of $(G, L_c^{A, \tilde{C}})$. Let F to be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $M_{\{i,k\}}(w)$. Let C' be the set of vertices $v \in \tilde{C} \setminus F$ with $N(v) \cap \tilde{A} \subseteq M_{\{i,k\}}(w)$. By construction, $c(v) \in L''(v)$ for all $L'' \in \mathcal{L}_{(i,j,k)}$ and $v \in V(G_{(i,j,k)}) \setminus (C' \cup F)$. Next, we show that the claim holds for every vertex in $C' \setminus D_{(i,j,k)}$. By (1) and construction, every vertex $c_1 \in C' \setminus D_{(i,j,k)}$ is adjacent to some vertex $c_2 \in \tilde{C} \setminus D_{(i,j,k)}$. By construction, both c_1 and c_2 have a neighbor in $M_{\{i,k\}}(w)$, hence belong to F , and so (4c) follows from 2.3. This proves (4).

A coloring c of (G, L, Z) is a *type II coloring* if there exists $w \in \tilde{C}$ with neighbors $x, y \in \tilde{A}_{\{1,2\}} \cup \tilde{A}_{\{1,3\}} \cup \tilde{A}_{\{2,3\}}$ such that $c(x) \neq c(y)$.

(5) Assume that (G, L, Z) does not have a type I coloring, and that no vertex of \tilde{C} has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$. If G has a type II coloring, then there exists a restriction \mathcal{P}_3 of (G, L, Z) of size $O(|V(G)|^7)$ such that:

(5a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$ and $X' = \mathcal{Z}$ for every $(G', L', X') \in \mathcal{P}_3$,

(5b) Every $(G', L', X') \in \mathcal{P}_3$ is such that $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and

(5c) \mathcal{P}_3 is colorable.

Moreover, \mathcal{P}_3 can be computed in time $O(|V(G)|^7)$.

If G does not have a type II coloring, then $\mathcal{P}_3 = \emptyset$.

Proof: Let $\{i, j, k\} = \{1, 2, 3\}$. If there does not exist a vertex $w \in \tilde{C}$ anticomplete to $\tilde{A}_{\{k\}}$ with $N_{\{i,k\}}(w) \neq \emptyset$ and $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \neq \emptyset$, set $\mathcal{P}_3 = \emptyset$ and halt. Otherwise, let $w \in \tilde{C}$ be anticomplete to $\tilde{A}_{\{k\}}$ with $N_{\{i,k\}}(w) \neq \emptyset$ and $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \neq \emptyset$. Since no vertex of \tilde{C} has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, it follows that either $N_{\{i,j\}}(w) = \emptyset$ or $N_{\{j,k\}}(w) = \emptyset$. Set

- $L'(w) = \{k\}$,
- $L'(v) = \{i\}$, for every $v \in N_{\{i,k\}}(w)$,
- $L'(v) = \{j\}$, for every $v \in N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w)$, and
- $L'(v) = L(v)$, for every $v \in V(G) \setminus (N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w) \cup N_{\{j,k\}}(w) \cup \{w\})$.

Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w) \cup N_{\{j,k\}}(w) \cup \{w\}$. First, update the palettes of the vertices in \tilde{A} with respect to A . Then, update the palettes of all the vertices in \tilde{C} with respect to \tilde{A} . And so after updating, for every $c \in \tilde{C}$ if $N(c) \cap A$ is non-empty, then $|L'(c)| \leq 2$. Let $v \in \tilde{C}$ with $|L'(v)| = 3$. Then, by the definition of updating, $N(v) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$. Let $F_{\{i,k\}}$ be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,k\}}$, and let $F_{\{j,k\}}$ be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $\tilde{A}_{\{j,k\}}$. Initialize $D_{(i,j,k)}^w = \emptyset$. Consider a vertex $c_1 \in \tilde{C} \setminus (F_{\{i,k\}} \cup F_{\{j,k\}})$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$ and $|L'(c_1)| = 3$. Recall that in every coloring of $(G \setminus c_1, L', \mathcal{Z})$ at most two colors appear in $N(c_1) \cap (M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w))$. Therefore, if c_1 is anticomplete to $\tilde{C} \setminus \{c_1\}$, then, by 3.1, it follows that $(G \setminus D_{(i,j,k)}^w, L'|_{V(G) \setminus D_{(i,j,k)}^w}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D_{(i,j,k)}^w \cup \{c_1\}), L'|_{V(G) \setminus (D_{(i,j,k)}^w \cup \{c_1\})}, \mathcal{Z})$ is colorable. In this case, set $D_{(i,j,k)}^w = D_{(i,j,k)}^w \cup \{c_1\}$. So we may assume c_1 is adjacent to some $c_2 \in \tilde{C}$. Suppose that c_1 is anticomplete to at least one of $M_{\{i,k\}}(w)$ and $M_{\{j,k\}}(w)$. Then in every coloring of $(G \setminus c_1, L', \mathcal{Z})$ at most one color appears in $N(c_1) \cap (M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w))$, and so, since $N(c_1) \subseteq \{c_2\} \cup M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$, we deduce that at most two colors appear in $N(c_1)$. Thus, again by 3.1, $(G \setminus D_{(i,j,k)}^w, L'|_{V(G) \setminus D_{(i,j,k)}^w}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D_{(i,j,k)}^w \cup \{c_1\}), L'|_{V(G) \setminus (D_{(i,j,k)}^w \cup \{c_1\})}, \mathcal{Z})$ is colorable. In this case, set $D_{(i,j,k)}^w = D_{(i,j,k)}^w \cup \{c_1\}$. Therefore we may assume that c_1 has both a neighbor in $M_{\{i,k\}}(w)$ and

a neighbor in $M_{\{j,k\}}(w)$. Since $c_1 \notin F_{\{i,k\}} \cup F_{\{j,k\}}$, it follows that c_2 is anticomplete to $\tilde{A}_{\{i,k\}} \cup \tilde{A}_{\{j,k\}}$. This implies that every neighbor of c_2 in $\tilde{A}_{\{i,j\}}$ either belongs to $N_{\{i,j\}}(w)$ or is complete to $N_{\{i,k\}}(w)$. This implies that $|L'(y)| = 1$ for every $y \in N(c_2) \setminus \{c_1\}$, and so, by the definition of updating, $L'(y) \subseteq \{1, 2, 3\} \setminus L(c_2)$ for all $y \in N(c_2) \setminus \{c_1\}$. If $|L'(c_2)| = 1$, set $L'(c_1) = L'(c_1) \setminus L'(c_2)$. Otherwise, $|L'(c_2)| = 2$ and so, by 3.2, it follows that $(G \setminus D_{(i,j,k)}^w, L'|_{V(G) \setminus D_{(i,j,k)}^w}, \mathcal{Z})$ is colorable if and only if $(G \setminus (D_{(i,j,k)}^w \cup \{c_1, c_2\}), L'|_{V(G) \setminus (D_{(i,j,k)}^w \cup \{c_1, c_2\})}, \mathcal{Z})$ is colorable. In this case, set $D_{(i,j,k)}^w = D_{(i,j,k)}^w \cup \{c_1, c_2\}$. Carry out this procedure for every $c_1 \in \tilde{C} \setminus (F_{\{i,k\}} \cup F_{\{j,k\}})$ with $N(c_1) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$ and $|L'(c_1)| = 3$. Let $G_{(i,j,k)}^w = G \setminus D_{(i,j,k)}^w$. Repeatedly applying the previous argument, it follows that (G, L', \mathcal{Z}) is colorable if and only if $(G_{(i,j,k)}^w, L'|_{V(G_{(i,j,k)}^w)}, \mathcal{Z})$ is colorable. By assumption, there exists $b \in \tilde{B}$ complete to $\tilde{A}_{\{i,k\}}$. Let \mathcal{L}' be the set of $O(|V(G)|^2)$ subpalettes of $L'|_{V(G_{(i,j,k)}^w)}$ obtained from 2.3 applied with

- $x = b$,
- $S = \tilde{A}_{\{i,k\}}$,
- $\hat{A} \cup \hat{B} = F_{\{i,k\}}$,
- $Y = V(G_{(i,j,k)}^w) \setminus (\{b\} \cup \tilde{A}_{\{i,k\}}(w) \cup F_{\{i,k\}})$, and
- $X = \mathcal{Z}$.

For each $\hat{L} \in \mathcal{L}'$, define the subpalette $L_{(i,j,k)}^w(\hat{L})$ of $L|_{V(G_{(i,j,k)}^w)}$ as follows: For $v \in V(G_{(i,j,k)}^w)$ set

$$L_{(i,j,k)}^w(\hat{L})(v) = \begin{cases} \hat{L}(v) & , \text{ if } v \in \tilde{C} \setminus D_{(i,j,k)}^w \\ L'(v) & , \text{ otherwise} \end{cases}$$

$$\mathcal{P}_{(i,j,k)}^w = \{(G_{(i,j,k)}^w, L_{(i,j,k)}^w(\hat{L}), \mathcal{Z}) : \hat{L} \in \mathcal{L}'\}.$$

By assumption, there exists $b' \in \tilde{B}$ complete to $\tilde{A}_{\{j,k\}}$. For each $\hat{L} \in \mathcal{L}'$, let $\mathcal{L}''(\hat{L})$ be the set of $O(|V(G)|^2)$ subpalettes of \hat{L} obtained from 2.3 applied with

- $x = b'$,
- $S = A_{\{j,k\}}$,
- $\hat{A} \cup \hat{B} = F_{\{j,k\}}$,
- $Y = V(G_{(i,j,k)}^w) \setminus (\{b'\} \cup A_{\{j,k\}}(w) \cup F_{\{j,k\}})$, and
- $X = \mathcal{Z}$.

Finally, for every $\hat{L} \in \mathcal{L}'$ and $L'' \in \mathcal{L}''(\hat{L})$, define the subpalette $L_{(i,j,k)}^w(\hat{L}, L'')$ of $L|_{V(G_{(i,j,k)})}$ as follows: For every $v \in V(G_{(i,j,k)}^w)$ set

$$L_{(i,j,k)}^w(\hat{L}, L'')(v) = \begin{cases} L''(v) & , \text{ if } v \in \tilde{C} \setminus D_{(i,j,k)}^w \\ \hat{L}(v) & , \text{ otherwise} \end{cases}$$

Let

$$\mathcal{P}_{(i,j,k)}^w = \{(G_{(i,j,k)}^w, L_{(i,j,k)}^w(\hat{L}, L''), \mathcal{Z}) : \hat{L} \in \mathcal{L}', L'' \in \mathcal{L}''(\hat{L})\}.$$

Let \mathcal{P}_3 be the union of all $\mathcal{P}_{(i,j,k)}^w$. Since there are at most $6n$ choices of w and $\{i, j, k\}$, and $F_{\{i,k\}}, F_{\{j,k\}}$ can be found in time $O(|V(G)|^2)$, it follows that building each $\mathcal{P}_{(i,j,k)}^w$ requires $O(|V(G)|^2)$ applications of 2.3, and so \mathcal{P}_3 can be constructed in time $O(|V(G)|^7)$. Now, we argue that \mathcal{P}_3 is indeed a restriction of (G, L, Z) . Suppose some $\mathcal{P}_{(i,j,k)}^w$ is colorable. By 2.3, it follows that $(G_{(i,j,k)}^w, L'|_{V(G_{(i,j,k)}^w)}, \mathcal{Z})$ is colorable, and so, as argued above, by 3.1 and 3.2, it follows that (G, L', \mathcal{Z}) is colorable. Since L' is a subpalette of L , and $Z \subseteq \mathcal{Z}$, we deduce that (G, L, Z) is colorable, and a coloring of (G, L, Z) can be reconstructed in linear time.

Suppose c is a type II coloring of (G, L, Z) . By (3), c is a coloring of (G, L, \mathcal{Z}) . By construction and 2.3, (5a) and (5b) hold. Next, we show that (5c) holds. Let $w \in \tilde{C}$ have neighbors of two different colors (under c) in $\tilde{A}_{\{1,2\}} \cup \tilde{A}_{\{1,3\}} \cup \tilde{A}_{\{2,3\}}$. By symmetry, we may assume $c(w) = k$ with $N_{\{i,k\}}(w) \neq \emptyset$. Then w is anticomplete to $\tilde{A}_{\{k\}}$. Since G admits a type II coloring and no type I coloring, we deduce that $N_{\{i,j\}}(w) \cup N_{\{j,k\}}(w) \neq \emptyset$. We claim that $\mathcal{P}_{(i,j,k)}^w$ is colorable. It follows that $c(v) = i$ for every $v \in N_{\{i,k\}}(w)$ and that $c(u) = j$ for every $u \in N_{\{j,k\}}(w)$. We claim that $c(v) = j$ for every $v \in N_{\{i,j\}}(w)$. Suppose not. Then $N_{\{i,j\}}(w) \neq \emptyset$ and $N_{\{j,k\}}(w) = \emptyset$. Since c is a type II coloring, it follows that there exists $y \in N_{\{i,j\}}(w)$ with $c(y) = j$. But since (G, L, Z) has no type I coloring, it follows that $c(u) = j$ for every $u \in N_{\{j,k\}}(w)$. This proves the claim.

Let $A = \tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}} \cup N_{\{i,j\}}(w) \cup N_{\{i,k\}}(w) \cup N_{\{j,k\}}(w) \cup \{w\}$. Let M be the palette of G obtained from L by first updating the palettes of the vertices in \tilde{A} with respect to A , and then updating the palettes of the vertices of \tilde{C} with respect to \tilde{A} . It follows that c is a coloring of (G, M, \mathcal{Z}) .

Let $F_{\{i,k\}}$ be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $\tilde{A}_{\{i,k\}}$, and let $F_{\{j,k\}}$ be the vertex set of the 2-vertex components of \tilde{C} such that both vertices of the component have a neighbor in $\tilde{A}_{\{j,k\}}$. Let C' be the set of vertices $v \in \tilde{C} \setminus (F_{\{i,k\}} \cup F_{\{j,k\}})$ with $N(v) \cap \tilde{A} \subseteq M_{\{i,k\}}(w) \cup M_{\{j,k\}}(w)$. By construction, $c(v) \in L''(v)$ for all $L'' \in \mathcal{L}_{\{i,j,k\}}^w$ and $v \in V(G_{\{i,j,k\}}^w) \setminus (C' \cup F_{\{i,k\}} \cup F_{\{j,k\}})$. Next, we consider vertices in $C' \setminus D_{(i,j,k)}^w$. By construction, every vertex $c_1 \in C' \setminus D_{(i,j,k)}^w$ has both a neighbor in $M_{\{i,k\}}(w)$, and a neighbor in $M_{\{j,k\}}(w)$, is adjacent to some vertex $c_2 \in \tilde{C}$, and c_2 has a neighbor in at least one of $\tilde{A}_{\{i,k\}}$ and $\tilde{A}_{\{j,k\}}$. Moreover, $|L_c^{\tilde{A}, \tilde{C}}(c_2)| = 1$,

and so, since $c(c_1) \neq c(c_2)$, it follows that $c(c_1) \in L''(c_1)$ for all $L'' \in \mathcal{L}_{(i,j,k)}^w$. Now $\mathcal{P}_{(i,j,k)}^w$ is colorable by 2.3, and (5c) follows. This proves (5).

(6) Assume (G, L, Z) does not have a type I or a type II coloring. Then there exists a restriction $\{(G', L', Z')\}$ of (G, L, Z) such that the following hold:

(6a) $\tilde{A} \cup \tilde{B} \subseteq V(G')$,

(6b) $L'(v) = L(v)$ for every $v \in \tilde{A} \cup \tilde{B}$, and $|L'(v)| \leq 2$ for every $v \in V(G') \cap \tilde{C}$, and

(6c) If (G, L, Z) is colorable, then (G', L', Z') is colorable.

Moreover, (G', L', Z') can be computed in time $O(|V(G)|^2)$.

Proof: Let C' be the set of vertices in \tilde{C} anticomplete to $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}}$. Let $G' = G \setminus C'$, let $W = \{N(v) \cap \tilde{A} : v \in C'\}$, and let $Z' = Z \cup W$. Update the palettes of all the vertices in $\tilde{C} \setminus C'$ with respect to $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}}$, and let L' be the palette $L|_{V(G')}$. Clearly, (6a) and (6b) hold. Suppose that (G, L, Z) is colorable. Since (G, L, Z) has no type I and no type II coloring, it follows that every set in Z' is monochromatic in every coloring of (G, L, Z) , and therefore (G', L', Z') is colorable. Thus (6c) holds.

Now, we argue that $\{(G', L', Z')\}$ is a restriction of (G, L, Z) . Let c be a coloring of (G', L', Z') ; we need to show that c can be extended to a coloring of (G, L, Z) in time $O(|V(G)|)$. Let $c_1 \in C'$. Then c_1 is anticomplete to $\tilde{A}_{\{1\}} \cup \tilde{A}_{\{2\}} \cup \tilde{A}_{\{3\}}$, and $|L(c_1)| = 3$. It follows that in every coloring of (G, L, Z) only one color appears in $N(c_1) \setminus \tilde{C}$. Recall also that c_1 has at most one neighbor in C . Now, repeatedly applying 3.1 to vertices of C' , it follows that c can be extended to a coloring of (G, L, Z) in time $O(|V(G)|)$. This prove (6).

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \{(G', L', Z')\}$ be the union of the sets of restrictions from (2), (4), (5) and (6). Then \mathcal{P} is a restriction of (G, L, Z) . We claim that \mathcal{P} satisfies the conclusion of the theorem. By (2), (4), (5) and (6), it follows that \mathcal{P} has size $O(|V(G)|^7)$, and that (a) and (b) hold. Next, we show that (c) holds. Since \mathcal{P} is a restriction of (G, L, Z) , by definition, it follows that if \mathcal{P} is colorable, then (G, L, Z) is colorable. By (2c), if (G, L, Z) has a type I coloring, then \mathcal{P}_1 is colorable. So we may assume (G, L, Z) does not have a type I coloring. By (4c), if some vertex of \tilde{C} has neighbors in all three of $\tilde{A}_{\{1,2\}}$, $\tilde{A}_{\{1,3\}}$ and $\tilde{A}_{\{2,3\}}$, then (G, L, Z) is colorable if and only if \mathcal{P}_2 is colorable. So we may assume no such vertex exists. By (5c), if (G, L, Z) has a type II coloring, then \mathcal{P}_3 is colorable. So we may assume (G, L, Z) does not have a type II coloring. By (6), it follows that (G, L, Z) is colorable if and only if (G', L', Z') is colorable. This proves 3.4. \square

4 Cleaning

In this section, we identify two configurations whose presence in a graph G allows us to delete some vertices and obtain an induced subgraph G' of G , such that G' is k -colorable if and only if G is k -colorable. Furthermore, these two configurations can be efficiently recognized, and so at the expense of carrying out a polynomial time procedure we may assume a given graph does not contain either configuration.

Let G be a graph. Recall that a vertex $v \in V(G)$ is *dominated* by a vertex $u \in V(G) \setminus \{v\}$ if u is non-adjacent to v and $N(v) \subseteq N(u)$. In terms of coloring, dominated vertices are useful because of the following:

4.1. *Let \mathcal{F} be a set of graphs and G be an \mathcal{F} -free graph. If $v \in V(G)$ is dominated by $u \in V(G) \setminus \{v\}$, then $G \setminus v$ is an \mathcal{F} -free graph which is k -colorable if and only if G is k -colorable. Furthermore, we can extend any k -coloring of $G \setminus v$ to a k -coloring of G in constant time.*

Proof. Clearly, $G \setminus v$ is \mathcal{F} -free, and if G is k -colorable, then $G \setminus v$ is k -colorable. Now, suppose c is a k -coloring of $G \setminus v$. Since v is non-adjacent to u and $N(v) \subseteq N(u)$, it follows, assigning $c(v) = c(u)$, that c extends to a coloring of G . This proves 4.1. \square

Let G be a graph. Recall that for a subset $X \subseteq V(G)$, we say that a vertex $v \in V(G) \setminus X$ is *mixed* on X , if v is not complete and not anticomplete to X . We say that a pair of disjoint non-empty stable sets (A, B) form a *non-trivial homogeneous pair of stable sets* if $2 < |A| + |B| < |V(G)|$ and no vertex $v \in V(G) \setminus (A \cup B)$ is mixed on A or mixed on B . In terms of coloring, non-trivial homogeneous pairs of stable sets are useful because of the following:

4.2. *Let \mathcal{F} be a set of graphs and let G be an \mathcal{F} -free graph. Let (A, B) be a non-trivial homogeneous pair of stable sets and choose $a \in A$ and $b \in B$, adjacent if possible. Then $G \setminus ((A \cup B) \setminus \{a, b\})$ is an \mathcal{F} -free graph which is k -colorable if and only if G is k -colorable. Furthermore, we can extend any k -coloring of $G \setminus ((A \cup B) \setminus \{a, b\})$ to a k -coloring of G in linear time.*

Proof. Clearly, $G \setminus ((A \cup B) \setminus \{a, b\})$ is \mathcal{F} -free, and if G is k -colorable, then $G \setminus ((A \cup B) \setminus \{a, b\})$ is k -colorable. Now, suppose c is a k -coloring of $G \setminus ((A \cup B) \setminus \{a, b\})$. By our choice of $a \in A$ and $b \in B$, if A is not anticomplete to B , then $c(a) \neq c(b)$. Since both A and B are stable and no vertex $v \in V(G) \setminus (A \cup B)$ is mixed on A or mixed on B , it follows, assigning $c(a') = c(a)$ for all $a' \in A$ and $c(b') = c(b)$ for all $b' \in B$, that c extends to a coloring of G . This proves 4.2. \square

We say a graph G is *clean* if G has no dominated vertices and no non-trivial homogeneous pair of stable sets.

4.3. *There is an algorithm with the following specifications:*

Input: *A graph G .*

Output: *A clean induced subgraph G' of G such that, for every integer k , G' is k -colorable if and only if G is k -colorable.*

Running time: $O(|V(G)|^5)$.

Furthermore, we can extend any k -coloring of G' to a k -coloring of G in linear time.

Proof. Since there are $O(|V(G)|^2)$ potential pairs of non-adjacent vertices $u, v \in V(G)$ and we can verify in time $O(|V(G)|)$ if $N(v) \subseteq N(u)$, it follows that in time $O(|V(G)|^3)$ we can find a dominated vertex in G , if one exists. In [8], King and Reed give an algorithm, that runs in time $O(|V(G)|^4)$, which is easily modified to produce a non-trivial homogeneous pair of stable sets, if one exists. And so, we can find a dominated vertex or a non-trivial homogeneous pair of stable sets in time $O(|V(G)|^4)$, if one exists. Hence, iteratively, 4.3 follows from the observations made in 4.1 and 4.2. \square

Thus, 4.3 implies that given a graph G , we may assume G is clean at the expense of carrying out a time $O(|V(G)|^5)$ procedure. It is also clear from 4.1 and 4.2, that given a graph G we may extend a k -coloring of the resulting clean induced subgraph G' of G produced by 4.3 to a k -coloring of G in time $O(|V(G)|)$.

5 A Useful Lemma

In this section we prove a general lemma which will be of great use when trying to apply 3.4 to clean graphs.

5.1. *Let G be a clean, connected $\{P_7, C_3\}$ -free graph with $V(G) = P \cup Q \cup R \cup S \cup T$ such that:*

- *$P \cup T$ is anticomplete to $R \cup S$,*
- *S is anticomplete to $P \cup Q$,*
- *every vertex in R has a neighbor in Q , and*
- *there exist $q_0 \in Q$ and $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q_0$ is an induced path.*
- *for every $q \in Q$, there exist $p_2, p_3 \in P$ and $p_1 \in P \cup \{q_0\}$ such that $p_1 - p_2 - p_3 - q$ is an induced path.*

Then the following hold:

1. S is empty
2. If for every $q \in Q$, there exist $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q$ is an induced path, then every component of R has size at most two.
3. Every component of R is bipartite, and if some component X of R has more than two vertices, then q_0 is complete to one side of the bipartition of $G[X]$.

Proof. (1) Let $q \in Q$. There is no induced path $q - r_1 - r_2 - r_3$ such that $r_1 \in R$ and $r_2, r_3 \in S$. Moreover, if there exist $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q$ is an induced path, then there is no induced path $q - r_1 - r_2 - r_3$ such that $r_1, r_2, r_3 \in R \cup S$.

Suppose there exist $p_1, p_2, p_3 \in P$ such that $p_1 - p_2 - p_3 - q$ is an induced path, and $q - r_1 - r_2 - r_3$ is an induced path with $r_1, r_2, r_3 \in R \cup S$. Then, since P is anticomplete to $R \cup S$, it follows that $p_1 - p_2 - p_3 - q - r_1 - r_2 - r_3$ is a P_7 in G , a contradiction. Next suppose that $r_1 \in R$ and $r_2, r_3 \in S$. There exist $p_1, p_2 \in P$ such that $q - p_1 - p_2 - q_0$ is an induced path. Since $q_0 - r_1 - r_2 - r_3$ is not an induced path in G by the previous argument, it follows that q_0 is anticomplete to $\{r_1, r_2, r_3\}$. But now $q_0 - p_1 - p_2 - q - r_1 - r_2 - r_3$ is an induced path in G , a contradiction. This proves (1).

(2) Every vertex in S has a neighbor in R .

Partition $S = S' \cup S''$, where S' is the set of vertices in S with a neighbor in R and $S'' = S \setminus S'$. Suppose S'' is non-empty. Since G is connected and S'' is anticomplete to $V(G) \setminus S$, it follows that there exists $s'' \in S''$ adjacent to $s' \in S'$. By definition, there exists $r \in R$ adjacent to s' and non-adjacent to s'' . Since every vertex in R has a neighbor in Q , there exists $q \in Q$ adjacent to r . Now $q - r - s' - s''$ is an induced path, contrary to (1). This proves (2).

(3) S is stable.

Suppose $s, s' \in S$ are adjacent. By (2), there exists $r \in R$ adjacent to s . Since G is triangle-free, s' is non-adjacent to r . Since every vertex in R has a neighbor in Q , there exists $q \in Q$ adjacent to r . However, then $q - r - s - s'$ is an induced path, contradicting (1). This proves (3).

Now, we prove 5.1.1. Consider a vertex $s \in S$. Since S is anticomplete to $P \cup Q$ and, by (3), S is stable, it follows that S is anticomplete to $V(G) \setminus R$. By (2), there exists $r \in R$ adjacent to s . Since every vertex in R has a neighbor in Q , there exists $q \in Q$ adjacent to r ; if possible, choose q and r such that there exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. Since s is not dominated by q , there exists $r' \in N(s) \setminus N(q)$. Since G is triangle-free, r' is non-adjacent to r , and it follows that $q - r - s - r'$ is an induced path. It follows from (1) there do not exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path, and in particular $q \neq q_0$. This implies that q_0 is anticomplete to $N(s)$,

and that there exist $p_1, p_2 \in P$ such that $q - p_1 - p_2 - q_0$ is an induced path. But now $q_0 - p_2 - p_1 - q - r - s - r'$ is a P_7 in G , a contradiction. This proves 5.1.1.

(4) *Every component of R is bipartite.*

Suppose X is a component of R which is not bipartite. Since G is $\{P_7, C_3\}$ -free, it follows that $G[X]$ contains either a C_5 or a C_7 . Let $x_1 - x_2 - \dots - x_{2k+1} - x_1$ be either a 5-gon or a 7-gon given by $x_1, \dots, x_{2k+1} \in R$ with $k \in \{2, 3\}$. Let $q \in Q$ be a vertex with a neighbor in $\{x_1, \dots, x_{2k+1}\}$. Since $2k+1$ is odd and G is triangle-free, we may assume that q is adjacent to x_1 and non-adjacent to x_2, x_3 . But now $q - x_1 - x_2 - x_3$ is an induced path, and so (1) implies that there do not exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. This implies that q_0 is anticomplete to $\{x_1, \dots, x_{2k+1}\}$, and that there exist $p_1, p_2 \in P$ such that $q - p_1 - p_2 - q_0$ is an induced path. But now $q_0 - p_2 - p_1 - q - x_1 - x_2 - x_3$ is a P_7 in G , a contradiction. This proves (4).

(5) *Let X be a component of R , and (X_1, X_2) be a bipartition of $G[X]$. Let $q \in Q$ be such that there exist $p_1, p_2, p_3 \in P$ where $q - p_1 - p_2 - p_3$ is an induced path. Then q is not mixed on either X_1 or X_2 .*

Suppose there exists a vertex $q \in Q$ adjacent to x and non-adjacent to x' , where $x, x' \in X_1$. Since X is connected, by choosing x and x' at minimum distance from each other in $G[X]$, we may assume that there exists $x_2 \in X_2$ adjacent to both x and x' . Since G is triangle-free, it follows that q is non-adjacent to x_2 . Now $q - x - x_2 - x'$ is an induced path, and so (1) implies that there do not exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. This proves (5).

Let X be a component of R , and (X_1, X_2) be a bipartition of $G[X]$. First we prove 5.1.2. Suppose that for every vertex of Q there exist $p_1, p_2, p_3 \in P$ such that $q - p_1 - p_2 - p_3$ is an induced path. By (5), no vertex of Q is mixed on X_1 , and similarly, no vertex of Q is mixed on X_2 . Since $P \cup T$ is anticomplete to R , it follows that $V(G) \setminus Q$ is anticomplete to R , and in particular to X . Hence, (X_1, X_2) is a homogeneous pair of stable sets, and so, since G is clean, it follows that $|X| \leq 2$. This proves 5.1.2.

Finally, we prove 5.1.3. Suppose that $|X| > 2$. Since (X_1, X_2) is not a homogeneous pair of stable sets, it follows that there exists a vertex $q \in Q$ adjacent to x and non-adjacent to x' , where $x, x' \in X_1$, say. Since X is connected, by choosing x and x' at minimum distance from each other in $G[X]$, we may assume that there exists $x_2 \in X_2$ adjacent to both x and x' . Since G is triangle-free, it follows that q is non-adjacent to x_2 . By (5), $q \neq q_0$, and there exist $p_1, p_2 \in P$ such that $q - p_1 - p_2 - q_0$ is a path. Since $q_0 - p_2 - p_1 - q - x - x_2 - x'$ is not a P_7 in G , it follows that q_0 has a neighbor in $\{x, x', x_2\}$, and 5.1.3 follows from (5). \square

6 7-gons

In this section we show that if a $\{P_7, C_3\}$ -free graph contains a 7-gon, then in polynomial time we can decide if the graph is 3-colorable, and give a coloring if one exists. Let C be an n -gon in a graph G . For a vertex $v \in V(G) \setminus V(C)$, we call the neighbors of v in $V(C)$ the *anchors of v in C* .

We begin with some definitions. Let C be a 7-gon in a graph G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. We say that a vertex $v \in V(G) \setminus V(C)$ is:

- a *clone at i* , if $N(v) \cap V(C) = \{v_{i-1}, v_{i+1}\}$ for some $i \in \{0, 1, \dots, 6\}$, where all indices are mod 7,
- a *propeller at $\{i, i+3\}$* , if $N(v) \cap V(C) = \{v_i, v_{i+3}\}$ for some $i \in \{0, 1, \dots, 6\}$, where all indices are mod 7,
- a *star at i* , if $N(v) \cap V(C) = \{v_{i-2}, v_i, v_{i+2}\}$ for some $i \in \{0, 1, \dots, 6\}$, where all indices are mod 7.

The following shows how we can partition the vertices of G based on their anchors in C .

6.1. *Let G be a $\{P_7, C_3\}$ -free graph, and suppose C is a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. If $v \in V(G) \setminus V(C)$, then for some $i \in \{0, 1, \dots, 6\}$ either:*

1. *v is a clone at i ,*
2. *v is a propeller at $\{i, i+3\}$,*
3. *v is a star at i , or*
4. *v is anticomplete to $V(C)$.*

Proof. Consider a vertex $v \in V(G) \setminus V(C)$. If v is anticomplete to $V(C)$, then 6.1.4 holds. Thus, we may assume $N(v) \cap V(C) \neq \emptyset$. Since G is triangle-free, and 7 is odd, we may assume that v is adjacent to v_0 , and anticomplete to $\{v_6, v_1, v_2\}$. If v is adjacent to v_4 , then, since G is triangle-free, 6.1.2 holds, so we may assume not. Since $v - v_0 - v_1 - v_2 - v_3 - v_4 - v_5$ is not a P_7 in G , it follows that v has a neighbor in $\{v_3, v_5\}$. If v is complete to $\{v_3, v_5\}$, then 6.1.3 holds; if v is adjacent to v_3 and not to v_5 , then 6.1.2 holds; and if v is adjacent to v_6 and not to v_3 , then 6.1.1 holds. This proves 6.1. \square

Let G be a triangle-free graph, and let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. Using 6.1 we partition $V(G) \setminus V(C)$ as follows:

- Let $CL^C(i)$ be the set of clones at i , and define $CL^C = \bigcup_{i=0}^6 CL^C(i)$.

- $P^C(i)$ be the set of propellers at $\{i, i+3\}$, and define $P^C = \bigcup_{i=0}^6 P^C(i)$.
- Let $S^C(i)$ be the set of stars at i and define $S^C = \bigcup_{i=0}^6 S^C(i)$.
- Let A^C be the set of vertices anticomplete to $V(C)$.

By 6.1, it follows that $V(G) = V(C) \cup CL^C \cup P^C \cup S^C \cup A^C$. Furthermore, we partition $A^C = X^C \cup Y^C \cup Z^C$, where

- X^C is the set of vertices in A^C with a neighbor in P^C ,
- Y^C is the set of vertices in $A^C \setminus X^C$ with a neighbor in S^C , and
- $Z^C = A^C \setminus (X^C \cup Y^C)$.

And so, for a given 7-gon C in time $O(|V(G)|^2)$ we obtain the partition $V(C) \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C \cup Z^C$ of $V(G)$. Now, we establish several properties of this partition.

6.2. *If G is a $\{P_7, C_3\}$ -free graph, then A^C is anticomplete to CL^C for every 7-gon C in G .*

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. Suppose there exists a vertex $u \in A^C$ adjacent to $v \in CL^C$. By symmetry, we may assume $v \in CL^C(0)$. However, then $u - v - v_1 - v_2 - v_3 - v_4 - v_5$ is a P_7 in G , a contradiction. This proves 6.2. \square

6.2 implies that:

6.3. *If G is a $\{P_7, C_3\}$ -free graph, then for every 7-gon C in G the following hold:*

1. A^C is anticomplete to $V(C) \cup CL^C$.
2. Every vertex in X^C has a neighbor in P^C (and possibly S^C).
3. $Y^C \cup Z^C$ is anticomplete to P^C .
4. Every vertex in Y^C has a neighbor in S^C .
5. Z^C is anticomplete to S^C .

For a fixed subset X of $V(G)$, we say a vertex $v \in V(G) \setminus X$ is *mixed on an edge of X* , if there exist adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$. We need the following two facts:

6.4. *If G is a $\{P_7, C_3\}$ -free graph, then for every 7-gon C in G the following hold:*

1. *No vertex in P^C is mixed on an edge of A^C .*
2. *X^C is stable and anticomplete to $Y^C \cup Z^C$.*

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$. Suppose for adjacent $a, a' \in A^C$, there exists $p \in P^C$ which is adjacent to a' and non-adjacent to a . By symmetry, we may assume $p \in P^C(0)$. However, then $a - a' - p - v_3 - v_4 - v_5 - v_6$ is a P_7 in G , a contradiction. This proves 6.4.1.

Consider a vertex $x \in X^C$. By 6.3.2, x has a neighbor $p \in P^C$. If there exists $x' \in N(x) \cap A^C$, then, since G is triangle-free, it follows that p is non-adjacent to x' , and so p is mixed on an edge of A^C , contradicting 6.4.1. This proves 6.4.2. □

6.5. *Let G be a clean, connected $\{P_7, C_3\}$ -free graph. Then for every 7-gon C in G the following hold:*

1. *Z^C is empty.*
2. *Every component of Y^C is a singleton or an edge.*

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$.

(1) *For every $s \in S^C$, there exists $t_1, t_2, t_3 \in V(C)$ such that $t_1 - t_2 - t_3 - s$ is an induced path.*

Consider a vertex $s \in S^C$. By symmetry, we may assume $s \in S^C(0)$. And so, $v_4 - v_3 - v_2 - s$ is the desired induced path. This proves (1).

By 6.3, 6.4.2 and (1), we may apply 5.1 letting $P = V(C)$, $Q = S^C$, $R = Y^C$, $S = Z^C$, and $T = CL^C \cup P^C \cup X^C$. Then 5.1.1 and 5.1.2 follow immediately from 5.1. This proves 6.5. □

Now, we prove the main result of the section.

6.6. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3\}$ -free graph G which contains a 7-gon.*

Output: *A 3-coloring of G , or a determination that none exists.*

Running time: $O(|V(G)|^{10})$.

Proof. Let C be a 7-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$, and observe that C can be found in time $O(|V(G)|^7)$. Since G is clean, by 6.5.1, it follows that Z^C is empty, and so we may partition $V(G) = V(C) \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C$ as usual. Next, fix a 3-coloring c of $G[V(C)]$. Define the order 3 palette L_c^C of G as follows:

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \\ \{1, 2, 3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $CL^C \cup P^C \cup S^C$ with respect to $V(C)$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$, while $|L_c^C(v)| = 3$ for all $v \in X^C \cup Y^C$. Observe that, by construction, (G, L_c^C) is colorable if and only the 3-coloring c of $G[V(C)]$ extends to a 3-coloring of G .

Let $A' = P^C \cup S^C$, and for every non-empty subset $X \subseteq \{1, 2, 3\}$, define $A'_X = \{a \in A' \text{ with } L_c^C(a) = X\}$.

(1) For every $u \in X^C \cup Y^C$ and $\{i, j, k\} = \{1, 2, 3\}$, $N(u) \cap A'_{\{i,j\}}$ is complete to $A'_{\{i,k\}} \setminus N(u)$.

It is enough to show that for every $x \in N(u) \cap A'_{\{i,j\}}$ and $y \in A'_{\{i,k\}} \setminus N(u)$, such that x is non-adjacent to y , there exists an induced 6-vertex path $x - p_1 - \dots - p_5$ with $p_1, \dots, p_5 \in V(C) \cup CL^C \cup \{y\}$. For if such a path exists, then, since, by 6.3.1, u is anticomplete to $V(C) \cup CL^C \cup \{y\}$, it follows that $u - x - p_1 - \dots - p_5$ is a P_7 in G , a contradiction.

Since $x \in A'_{\{i,j\}}$ and $y \in A'_{\{i,k\}}$, it follows from the definition of updating that all the anchors of x are colored k , and all the anchors of y are colored j . In particular, this implies that x and y have no anchors in common.

Let $\{a, b\} = \{x, y\}$. First, suppose a is a star. By symmetry, we may assume $a \in S^C(0)$. Since a and b have no anchors in common, it follows that b is anticomplete to $\{v_0, v_2, v_5\}$. Since $b \in P^C \cup S^C$, it follows that $|N(b) \cap (V(C) \setminus \{v_0, v_2, v_5\})| \geq 2$ and so, by symmetry and 6.1, we may assume that v_1 is an anchor of b , that is, that $b \in P^C(1) \cup S^C(1) \cup S^C(6)$. Further by symmetry, we may assume that $b \in P^C(1) \cup S^C(1)$. Suppose $a = x$. If $b \in P^C(1)$, then $x - v_0 - v_1 - y - v_4 - v_3$ is the desired path, and if $b \in S^C(1)$, then $x - v_0 - v_1 - y - v_3 - v_4$ is the desired path. Thus, we may assume $b = x$. If $b \in P^C(1)$, then $x - v_4 - v_3 - v_2 - y - v_0$ is the desired path, and if $b \in S^C(1)$, then $x - v_3 - v_4 - v_5 - y - v_0$ is the desired path. Hence, we may assume neither of x, y is a star, that is, that both x and y are propellers. By symmetry, we may assume $x \in P^C(0)$, and therefore y is anticomplete to $\{v_0, v_3\}$. Since $y \in P^C$, it follows that $|N(y) \cap (V(C) \setminus \{v_0, v_3\})| = 2$ and so, by symmetry, we may assume that v_1 is an anchor of y , that is, that $y \in P^C(1) \cup P^C(5)$. If $y \in P^C(1)$, then $x - v_0 - v_1 - y - v_4 - v_5$ is the desired path, and if $y \in P^C(5)$, then $x - v_3 - v_2 - v_1 - y - v_5$ is the desired path. By our initial observation, this proves (1).

(2) For all distinct $i, j \in \{1, 2, 3\}$ some vertex of $V(C) \cup CL^C$ is complete to $A'_{\{i,j\}}$.

If $A'_{\{i,j\}} = \emptyset$, then (2) trivially holds. Thus, we may assume $A'_{\{i,j\}} \neq \emptyset$. Let $\{i, j, k\} = \{1, 2, 3\}$ and define K to be the set of vertices of $V(C)$ with a neighbor in $A'_{\{i,j\}}$. Since we have updated, it follows that $c(v) = k$ for every $v \in K$. Since $G[V(C)]$ has no stable set of size 4, it follows that $|K| \leq 3$. Since $A'_{\{i,j\}} \subseteq P^C \cup S^C$, it follows that $|K| \geq 2$. If $|K| = 2$, then $A_{\{i,j\}} \subseteq P^C$, and it follows that K is complete to $A'_{\{i,j\}}$. Hence, we may assume $|K| = 3$. By symmetry, we may assume that $K = \{v_0, v_3, v_5\}$. Since G is triangle-free, it follows that $A'_{\{i,j\}} \subseteq P^C(0) \cup S^C(5)$, and so v_3 is complete to $A_{\{i,j\}}$. This proves (2).

By 6.4.2 and 6.5, it follows that every component of $X^C \cup Y^C$ has at most two vertices. And so, by 6.3, (1) and (2), we can apply 3.4 with

- $\tilde{A} = A'$,
- $\tilde{B} = V(C) \cup CL^C$,
- $\tilde{C} = X^C \cup Y^C$, and
- $Z = \emptyset$.

Let \mathcal{P}_c^C be the restriction of (G, L_c^C, \emptyset) , of size $O(|V(G)|^7)$, thus obtained. By 3.4, \mathcal{P}_c^C can be computed in time $O(|V(G)|^7)$. By 3.4(c), we have that (G, L_c^C, \emptyset) (and, equivalently, (G, L_c^C)) is colorable if and only if \mathcal{P}_c^C is colorable. Consider $(G', L', X') \in \mathcal{P}_c^C$. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$, by 3.4(b), it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Thus, since $|X'|$ has size $O(|V(G)|)$, applying 1.7, we can test in time $O(|V(G)|^3)$ if (G', L', X') is colorable, and extend the coloring to (G, L_c^C) . Consequently, via $O(|V(G)|^7)$ applications of 1.7, we can determine if \mathcal{P}_c^C is colorable and extend any coloring of a colorable $(G', L', X') \in \mathcal{P}_c^C$ to a coloring of G . That is, in time $O(|V(G)|^{10})$ we can determine if the 3-coloring c of $G[V(C)]$ extends to a 3-coloring of G , and give an explicit 3-coloring c' of G such that $c'(v) = c(v)$ for all $v \in V(C)$, if one exists. Finally, let \mathcal{P} be the union of \mathcal{P}_c^C taken over all 3-colorings c of $G[V(C)]$. Since there are at most 7^3 3-colorings of $G[V(C)]$, it follows that we can test in time $O(|V(G)|^{10})$ if \mathcal{P} is colorable. Since every 3-coloring of G restricts to a 3-coloring of $G[V(C)]$, it follows that G is 3-colorable if and only if \mathcal{P} is colorable. This proves 6.6. □

7 Shells

We remind the reader, that a shell in a graph G is a pair (C, p) , where C is a 6-gon given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p \in V(G) \setminus \{v_0, \dots, v_5\}$, such that $N(p) \cap \{v_0, \dots, v_5\} = \{v_\ell, v_{\ell+3}\}$ for some $\ell \in \{0, 1, 2\}$. In this section we show that if a $\{P_7, C_3, C_7\}$ -free graph contains a shell, then in polynomial time we can decide if the graph is 3-colorable, and give a coloring if one exists.

We begin with some definitions. Let C be a 6-gon in a graph G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We say that a vertex $v \in V(G) \setminus V(C)$ is:

- a *leaf at i* , if $N(v) \cap V(C) = \{v_i\}$ for some $i \in \{0, 1, \dots, 5\}$,
- a *clone at i* , if $N(v) \cap V(C) = \{v_{i-1}, v_{i+1}\}$ for some $i \in \{0, 1, \dots, 5\}$, where all indices are mod 6,
- a *propeller at $\{i, i+3\}$* , if $N(v) \cap V(C) = \{v_i, v_{i+3}\}$ for some $i \in \{0, 1, \dots, 5\}$, where all indices are mod 6,
- an *even star*, if $N(v) \cap V(C) = \{v_0, v_2, v_4\}$,
- an *odd star*, if $N(v) \cap V(C) = \{v_1, v_3, v_5\}$.

The following shows how we can partition the vertices of G based on their anchors in C .

7.1. *Let G be a triangle-free graph, and suppose C is a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. If $v \in V(G) \setminus V(C)$, then for some $i \in \{0, 1, \dots, 5\}$ either:*

1. *v is a leaf at i ,*
2. *v is a clone at i ,*
3. *v is a propeller at $\{i, i+3\}$, where all indices are mod 6,*
4. *v is an even star,*
5. *v is an odd star, or*
6. *v is anticomplete to $V(C)$.*

Proof. Consider a vertex $v \in V(G) \setminus V(C)$. If v is anticomplete to $V(C)$, then 7.1.6 holds. Thus, we may assume $N(v) \cap V(C) \neq \emptyset$. By symmetry, suppose that $v_0 \in N(v) \cap V(C)$. Since G is triangle-free, it follows that v is anticomplete to $\{v_1, v_5\}$. Suppose v is non-adjacent to v_3 . If v is anticomplete to $\{v_2, v_4\}$, then 7.1.1 holds. If v is mixed on $\{v_2, v_4\}$, then 7.1.2 holds. If v is complete to $\{v_2, v_4\}$, then 7.1.4 holds. Thus, we may assume v is adjacent to v_3 . Since G is triangle-free, it follows that v is anticomplete to $\{v_2, v_4\}$, and so 7.1.3 holds. This proves 7.1. □

Let G be a triangle-free graph and C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We partition $V(G) \setminus V(C)$ as follows:

- Let $M^C(i)$ be the set of leaves at i and define $M^C = \bigcup_{i=0}^5 M^C(i)$.

- Let $CL^C(i)$ be the set of clones at i and define $CL^C = \bigcup_{i=0}^5 CL^C(i)$.
- Let $P^C(\{i, i+3\})$ be the set of propellers at $\{i, i+3\}$ and define $P^C = \bigcup_{i=0}^5 P^C(\{i, i+3\})$.
- Let S_0^C be the set of even stars.
- Let S_1^C be the set of odd stars.
- Let $S^C = S_0^C \cup S_1^C$.
- Let A^C be the set of vertices anticomplete to $V(C)$.

By 7.1, it follows that $V(G) = V(C) \cup M^C \cup CL^C \cup P^C \cup S^C \cup A^C$. Furthermore, we partition $A^C = X^C \cup Y^C \cup Z^C$, where

- X^C is the set of vertices in A^C with a neighbor in CL^C ,
- Y^C is the set of vertices in $A^C \setminus X^C$ with a neighbor in P^C ,
- $Z^C = A^C \setminus (X^C \cup Y^C)$.

And so, given a 6-gon C in G in time $O(|V(G)|^2)$ we obtain the partition $V(C) \cup M^C \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C \cup Z^C$ of $V(G)$. Now, we establish several properties of this partition. The following is immediate:

7.2. *If G is a triangle-free graph, then for every 6-gon C in G the following hold:*

1. *Every vertex in X^C has a neighbor in CL^C .*
2. *Every vertex in Y^C has a neighbor in P^C .*
3. *CL^C is anticomplete to $Y^C \cup Z^C$.*

Next, we show:

7.3. *If G is a $\{P_7, C_3, C_7\}$ -free graph, then for every 6-gon C in G the following hold:*

1. *M^C is anticomplete to A^C .*
2. *For every $q \in M^C \cup CL^C \cup P^C$, there exists $p_1, p_2, p_3 \in V(C)$ such that $p_1 - p_2 - p_3 - q$ is an induced path.*
3. *X^C is stable and anticomplete to $Y^C \cup Z^C$.*

4. Z^C is anticomplete to $V(G) \setminus (Y^C \cup S^C)$.
5. For every $i \in \{0, \dots, 5\}$, if $M^C(i)$ is non-empty, then $M^C(i+2) \cup M^C(i-2)$ is empty, where all indices are mod 6.
6. No vertex in A^C has a neighbor in $CL^C(i)$ and $CL^C(j)$ for $i \neq j$.

Proof. Let C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. Suppose there exists $a \in A^C$ adjacent to $m \in M^C$. By symmetry, we may assume $m \in M^C(0)$. However, then $a - m - v_0 - v_1 - v_2 - v_3 - v_4$ is a P_7 in G , a contradiction. This proves 7.3.1.

Consider a vertex $q \in M^C \cup CL^C \cup P^C$. By symmetry, we may assume q is adjacent to v_0 and non-adjacent to v_1, v_2 , and so $v_2 - v_1 - v_0 - q$ is an induced path. This proves 7.3.2.

Consider a vertex $x \in X^C$. By 7.2.1, there exists $c \in CL^C$ adjacent to x . By symmetry, we may assume $c \in CL^C(0)$. Let C' be the 6-gon given by $c - v_1 - v_2 - v_3 - v_4 - v_5 - c$. Suppose there exists $x' \in A^C$ adjacent to x . Since G is triangle-free, it follows that c is non-adjacent to x' . However, then $x \in M^{C'}$ is adjacent to $x' \in A^{C'}$, contrary to 7.3.1. This prove 7.3.3.

By 7.2.3 and 7.3.1, it follows that 7.3.4 holds.

To prove 7.2.5, suppose there exists $m \in M^C(0)$ and $m' \in M^C(2)$. If m is non-adjacent to m' , then $m' - v_2 - v_3 - v_4 - v_5 - v_0 - m$ is a P_7 in G , and if m is adjacent to m' , then $m - m' - v_2 - v_3 - v_4 - v_5 - v_0 - m$ is a C_7 in G , in both cases a contradiction. This proves 7.3.5.

We now prove 7.3.6. Assume $a \in A^C$ is adjacent to $c \in CL^C$. By symmetry, we may assume $c \in CL^C(0)$. Suppose there exists $c' \in CL^C \setminus CL^C(0)$ adjacent to a . Since G is triangle-free, it follows that c is non-adjacent to c' . By symmetry, it suffices to consider $c' \in CL^C(1) \cup CL^C(2) \cup CL^C(3)$. If $c' \in CL^C(1)$, then $a - c' - v_2 - v_3 - v_4 - v_5 - c - a$ is a C_7 in G , if $c' \in CL^C(2)$, then $v_2 - v_3 - c' - a - c - v_5 - v_0$ is a P_7 in G , and if $c' \in CL^C(3)$, then $v_0 - v_1 - c - a - c' - v_4 - v_3$ is a P_7 in G , in all three cases a contradiction. This proves 7.3.6. This proves 7.3. □

Let C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We say that a graph G has a *type I coloring with respect to C* if there exists a 3-coloring c of G such that $c(v_i) = c(v_{i+3})$ for every $i \in \{0, 1, 2\}$.

7.4. *There is an algorithm with the following specifications:*

Input: A clean, connected $\{P_7, C_3, C_7\}$ -free graph G .

Output: A type I coloring of G with respect to some 6-gon in G , or a determination that none exists.

Running time: $O(|V(G)|^{16})$.

Proof. In time $O(|V(G)|^6)$, we can enumerate all 6-gons in G . If G is C_6 -free, then clearly G does not have a type I coloring and we may halt. Hence, we may assume there exists a 6-gon C in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. In time $O(|V(G)|^2)$, we can partition $V(G) = V(C) \cup M^C \cup CL^C \cup P^C \cup S^C \cup X^C \cup Y^C \cup Z^C$ as usual. If S^C is non-empty, then G does not have a type I coloring with respect to C , since, by definition, the anchors of any star in C receive three distinct colors in any type I coloring. Hence, we may assume S^C is empty. Next, fix a 3-coloring c of $G[V(C)]$ such that $c(v_i) = c(v_{i+3})$ for every $i \in \{1, 2, 3\}$, where all indices are mod 6. Define the order 3 palette L_c^C of G as follows: For every $v \in V(G)$, set

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \\ \{1, 2, 3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $M^C \cup CL^C \cup P^C$ with respect to $V(C)$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C \cup Z^C)$, while $|L_c^C(v)| = 3$ for all $v \in X^C \cup Y^C \cup Z^C$. Additionally, $|L_c^C(v)| = 2$ if and only if $v \in M^C \cup P^C$. Observe that, by construction, (G, L_c^C) is colorable if and only if the 3-coloring c of $G[V(C)]$ extends to a type I coloring of G .

By 7.2, 7.3.1, 7.3.2 and 7.3.4, we may apply 5.1 letting $P = V(C) \cup M^C$, $Q = CL^C \cup P^C$, $R = X^C \cup Y^C$, $S = Z^C$, and $T = \emptyset$. It follows that Z^C is empty and that every component of $X^C \cup Y^C$ has size at most two. Let $A' = CL^C \cup P^C$, and for every non-empty subset $X \subseteq \{1, 2, 3\}$, define $A'_X = \{a \in A' \text{ with } L_c^C(a) = X\}$. Since for $v \in A'$, $|L_c^C(v)| = 2$ if and only if $v \in P^C$, it follows that if $|X| = 2$, then $A'_X = P^C(\{i, i+3\})$ for some $i \in \{0, 1, 2\}$.

(1) For every $c_1, c_2 \in X^C \cup Y^C$ and $\{i, j, k\} = \{1, 2, 3\}$, $(N(c_1) \cap A'_{\{i,j\}}) \setminus N(c_2)$ is complete to $(N(c_2) \cap A'_{\{i,k\}}) \setminus N(c_1)$.

We may assume $A'_{\{i,j\}} = P^C(\{0, 3\})$ and $A'_{\{i,k\}} = P^C(\{1, 4\})$. Suppose there exists $p_1 \in P^C(\{0, 3\}) \setminus N(c_2)$ adjacent to c_1 and $p_2 \in P^C(\{1, 4\}) \setminus N(c_1)$ adjacent to c_2 such that p_1 is non-adjacent to p_2 . If c_1 is non-adjacent to c_2 , then $c_2 - p_2 - v_1 - v_2 - v_3 - p_1 - c_1$ is a P_7 in G , and if c_1 is adjacent to c_2 , then $c_1 - c_2 - p_2 - v_1 - v_2 - v_3 - p_1 - c_1$ is a C_7 in G , in both cases a contradiction. Hence, it follows that p_1 is adjacent to p_2 . This proves (1).

(2) For all distinct $i, j \in \{1, 2, 3\}$ some vertex of $V(C) \cup M^C$ is complete to $A'_{\{i,j\}}$.

After updating, it follows that $|L_c^C(v)| = 2$ if and only if $v \in M^C \cup P^C$. By symmetry, we may assume $A'_{\{i,j\}} = P^C(\{0, 3\})$. Hence, $\{v_0, v_3\}$ is complete to $\tilde{A}_{\{i,j\}}$. This proves (2).

By (1) and (2), we can apply 3.4 with

- $\tilde{A} = CL^C \cup P^C$,
- $\tilde{B} = V(C) \cup M^C$,

- $\tilde{C} = X^C \cup Y^C$, and
- $Z = \emptyset$.

Let \mathcal{P}_c^C be the restriction of (G, L_c^C, \emptyset) of size $O(|V(G)|^7)$ thus obtained. By 3.4, \mathcal{P}_c^C can be computed in time $O(|V(G)|^7)$. By 3.4(c), we have that (G, L_c^C, \emptyset) (and equivalently (G, L_c^C)) is colorable if and only if \mathcal{P}_c^C is colorable. Consider $(G', L', X') \in \mathcal{P}_c^C$. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$, by 3.4(b), it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Since $|X'|$ has size $O(|V(G)|)$, applying 1.7, it follows that we can determine in time $O(|V(G)|^3)$ if (G', L', X') is colorable, and if it is, extend the coloring to a coloring of G . Therefore, via $O(|V(G)|^7)$ applications of 1.7, we can determine if \mathcal{P}_c^C is colorable. That is, in time $O(|V(G)|^{10})$ we can determine if the 3-coloring c of $G[V(C)]$ extends to a type I coloring of G , and give an explicit type I coloring c' of G such that $c'(v) = c(v)$ for all $v \in V(C)$, if one exists. Finally, let \mathcal{P} be the union of \mathcal{P}_c^C taken over all 6-gons C in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$ with S^C empty and all 3-colorings c of $G[V(C)]$ such that $c(v_i) = c(v_{i+3})$ for every $i \in \{1, 2, 3\}$, where all indices are mod 6. Since every type I coloring of G restricts to such a coloring of $G[V(C)]$, it follows that G has a type I coloring if and only if \mathcal{P} is colorable. Since there are $O(|V(G)|^6)$ 6-gons in G and $3!$ such colorings of $G[V(C)]$, it follows that \mathcal{P} consists of $O(|V(G)|^6)$ restrictions \mathcal{P}_c^C , and so by the previous argument, we can determine in time $O(|V(G)|^{16})$ if G admits a type I coloring, and construct such a coloring if one exists. This proves 7.4. \square

Let C be a 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. We say that a graph G has a *type II coloring with respect to C* (or just a *type II coloring* when the details are not important) if there exist a 3-coloring c of G such that $c(p_1) \neq c(p_2)$ for some $p_1, p_2 \in P^C(0, 3)$.

7.5. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3, C_7\}$ -free graph G , that does not admit a type I coloring.*

Output: *A type II coloring of G with respect to some 6-gon in G , or a determination that none exists.*

Running time: $O(|V(G)|^{18})$.

Proof. In time $O(|V(G)|^8)$, we can enumerate all triples (C, p_1, p_2) in G , where C is a 6-gon given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p_1, p_2 \in P^C(0, 3)$. If G has no such triple, then clearly G does not have a type II coloring and we may halt. Hence, we may assume there exists a 6-gon C in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p_1, p_2 \in P^C(0, 3)$. In time $O(|V(G)|^2)$, we can partition $V(G) = V(C) \cup M^C \cup CL^C \cup PC \cup S^C \cup X^C \cup Y^C \cup Z^C$ as usual. Write $D = V(C) \cup \{p_1, p_2\}$. Fix a 3-coloring c of $G[D]$ such that $c(p_1) \neq c(p_2)$. Since G does not admit a type I coloring, we may assume that $c(v_0) = c(v_3) = 1, c(v_1) =$

$c(v_5) = 2, c(v_2) = c(v_4) = 3, c(p_1) = 2$, and $c(p_2) = 3$. Define the order 3 palette L_c^C of G as follows: For every $v \in V(G)$, set

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in D \\ \{1, 2, 3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $M^C \cup CL^C \cup (P^C \setminus \{p_1, p_2\}) \cup S^C$ with respect to D . And so, $|L_c^C(v)| \leq 2$ if and only if $v \in V(G) \setminus (X^C \cup Y^C \cup Z^C)$. Moreover, for $v \in V(G) \setminus (X^C \cup Y^C \cup Z^C)$, $|L(v)| = 2$ only if $v \in M^C \cup CL^C(0) \cup CL^C(3) \cup P^C(0, 3)$. By construction, (G, L_c^C) is colorable if and only if the 3-coloring c of $G[D]$ extends to a type II coloring of G .

Observe that for every $v \in S^C(0)$ and $i \in \{1, 2\}$, $v - v_2 - v_3 - p_i$ is an induced path in G , and for every $v \in S^C(1)$ and $i \in \{1, 2\}$, $v - v_1 - v_0 - p_i$ is an induced path in G . Let W^C be the vertices of Z^C with a neighbor in S^C . Now by 7.3.1 and 7.3.2, we may apply 5.1 letting $P = V(C)$, $Q = M^C \cup CL^C \cup P^C \cup S^C$, $R = X^C \cup Y^C \cup W^C$, $S = Z^C \setminus W^C$, $T = \emptyset$, and $q_0 = p_1$. It follows that $Z^C \setminus W^C = \emptyset$, that every component of R is bipartite, and if some component of R has more than two vertices, then p_1 is complete to at least one side of the bipartition. Symmetrically, if some component of R has more than two vertices, then p_2 is complete to at least one side of the bipartition.

Let R_1 be the union of the components of R that contain a vertex complete to $\{p_1, p_2\}$. For every $v \in R_1$ that is complete to $\{p_1, p_2\}$, set $L_c^C(v) = \{1\}$. For every component X of R_1 with $|X| > 1$, proceed as follows. Let (A, B) be a bipartition of $G[X]$. By 5.1 and the definition of R_1 , it follows that one of A, B is complete to $\{p_1, p_2\}$. We may assume that A is complete to $\{p_1, p_2\}$. Therefore $L_c^C(a) = \{1\}$ for every $a \in A$. Now set $L_c^C(b) = \{2, 3\}$ for every $b \in B$. Note that this does not change the colorability of (G, L_c^C) . Observe that at this stage $|L_c^C(v)| < 3$ for every $v \in R_1$, and $|L_c^C(v)| = 3$ for every $v \in R \setminus R_1$.

Let R_2 be the union of all components Y of $R \setminus R_1$ such that $Y = \{y\}$ and y has a neighbor in $\{p_1, p_2\}$. For every $v \in R_2$, update the list of v with respect to $\{p_1, p_2\}$.

(1) Let $v \in CL^C(0) \cup CL^C(3)$ be adjacent to $y \in R$. Then each of p_1, p_2 has a neighbor in $\{v, y\}$.

Suppose not. We may assume that $v \in CL^C(0)$. If p_1 is anticomplete to $\{v, y\}$, then $y - v - v_1 - v_0 - p_1 - v_3 - v_4$ is a P_7 in G . This proves that either v or y is adjacent to p_1 . Similarly, either v or y is adjacent to p_2 . This proves (1).

Let $C' = R \setminus (R_1 \cup R_2)$. Then $|L_c^C(y)| = 3$ for every $y \in C'$. Moreover, no vertex of C' is complete to $\{p_1, p_2\}$, and if Y is a component of C' with $|Y| = 1$, then Y is anticomplete to $\{p_1, p_2\}$. Let A' be the set of vertices of $M^C \cup CL^C \cup P^C \cup S^C$ with a neighbor in C' . For every non-empty subset $X \subseteq \{1, 2, 3\}$, define $A'_X = \{a \in A' \text{ with } L_c^C(a) = X\}$.

Suppose that $v \in A' \cap (CL^C(0) \cup CL^C(3))$ has a neighbor y in C' . By 7.3.3, $\{y\}$ is a component of R . It follows from the definition of C' that y is anticomplete to $\{p_1, p_2\}$. Now (1) implies that v is complete to $\{p_1, p_2\}$, and, in particular, $L_c^C(v) = \{1\}$.

Consequently, for $v \in A'$, $|L_c^C(v)| = 2$ if and only if $v \in P^C(0, 3)$ and $L_c^C(v) = \{2, 3\}$. Thus $A'_{\{1,2\}} = A'_{\{1,2\}} = \emptyset$, and v_0 is complete to $A_{\{2,3\}}$.

We apply 3.4 with

- $\tilde{A} = A'$,
- $\tilde{B} = V(G) \setminus (A' \cup C')$,
- $\tilde{C} = C'$, and
- $Z = \emptyset$.

Let $\mathcal{P}_c^{C,p_1,p_2}$ be the restriction of (G, L_c^C, \emptyset) of size $O(|V(G)|^7)$ thus obtained. By 3.4, $\mathcal{P}_c^{C,p_1,p_2}$ can be computed in time $O(|V(G)|^7)$. By 3.4(c), we have that (G, L_c^C, \emptyset) (and equivalently (G, L_c^C)) is colorable if and only if $\mathcal{P}_c^{C,p_1,p_2}$ is colorable. Consider $(G', L', X') \in \mathcal{P}_c^{C,p_1,p_2}$. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus V(C')$, by 3.4(b), it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Since X' has size $O(|V(G)|)$, applying 1.7, it follows that we can test in time $O(|V(G)|^3)$ if (G', L', X') is colorable, and extend the coloring to (G, L_c^C) if it is. Therefore, via $O(|V(G)|^7)$ applications of 1.7, we can determine in time $O(|V(G)|^{10})$ if the 3-coloring c of $G[D]$ extends to a type II coloring of G , and give an explicit type II coloring c' of G such that $c'(v) = c(v)$ for all $v \in D$, if one exists. Finally, let \mathcal{P} be the union of $\mathcal{P}_c^{C,p_1,p_2}$ taken over all triples (C, p_1, p_2) where C is a 6-gon given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and $p_1, p_2 \in P^C(0, 3)$, and all 3-colorings c of $G[V(C) \cup \{p_1, p_2\}]$ such that $c(p_1) \neq c(p_2)$. Since every type II coloring of G restricts to such a coloring of $G[V(C) \cup \{p_1, p_2\}]$ for some C, p_1, p_2 , it follows that G has a type II coloring if and only if \mathcal{P} is colorable. Since there are $O(|V(G)|^8)$ such triples (C, p_1, p_2) in G and 2 such colorings of $G[V(C) \cup \{p_1, p_2\}]$ for each (C, p_1, p_2) , the restriction \mathcal{P} is the union of $O(|V(G)|^8)$ restrictions $\mathcal{P}_c^{C,p_1,p_2}$. Therefore by the previous argument, we can determine in time $O(|V(G)|^{18})$ if G admits a type II coloring, and construct such a coloring if one exists. This proves 7.5. \square

Now, suppose (C, p) is a shell in G . We partition $V(G) \setminus (V(C) \cup \{p\})$ as follows:

- Let Q_p^C be the set of vertices in $V(G) \setminus (V(C) \cup \{p\})$ with a neighbor in $V(C) \cup \{p\}$.
- Let R_p^C be the set of vertices in $V(G) \setminus (V(C) \cup \{p\} \cup Q_p^C)$ with a neighbor in Q_p^C .
- Let $S_p^C = V(G) \setminus (V(C) \cup \{p\} \cup Q_p^C \cup R_p^C)$.
- Let PL_p^C be the set of vertices in $V(G) \setminus V(C)$ adjacent to p and anticomplete to $V(C)$. Note, that PL_p^C is a subset of Q_p^C .

7.6. *Let G be a clean, connected $\{P_7, C_3, C_7\}$ -free graph. Then for every shell (C, p) in G the following hold:*

1. $M^C \cup CL^C \cup P^C \cup S^C \cup PL_p^C$ gives a partition of Q_p^C .
2. Every vertex of Q_p^C with a neighbor in R_p^C either belongs to PL_p^C or has at least two neighbors in $V(C)$.
3. S_p^C is empty.
4. Every component of R_p^C has size at most two.

Proof. Let (C, p) be a shell in G , where C is the 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$ and $p \in P^C$. By 7.1, it follows that 7.6.1 holds. Since R_p^C is a subset of A^C , by 7.3.1, it follows that M^C is anticomplete to R_p^C . And so, by definition, 7.1, and 7.6.1, it follows that 7.6.2 holds.

(1) For every $s \in S^C$, there exists $p_1, p_2, p_3 \in V(C) \cup \{p\}$ such that $p_1 - p_2 - p_3 - s$ is an induced path.

Consider a vertex $s \in S^C$. By symmetry, we may assume both s and p are adjacent to v_0 , that is, that $s \in S_0^C$ and $p \in P^C(\{0, 3\})$. Since G is triangle-free, it follows that s is non-adjacent to p . Then $v_3 - p - v_0 - s$ is the desired induced path. This proves (1).

By definition, S_p^C is anticomplete to $V(C) \cup \{p\} \cup Q_p^C$. Since G is clean and connected, by 7.3.2 and (1), we may apply 5.1 letting $P = V(C) \cup \{p\}$, $Q = Q_p^C$, $R = R_p^C$, $S = S_p^C$ and $T = \emptyset$. It follows that 7.6.3 and 7.6.4 hold. This proves 7.6. □

7.7. *There is an algorithm with the following specifications:*

Input: A clean, connected $\{P_7, C_3, C_7\}$ -free graph G which contains a shell.

Output: A 3-coloring of G , or a determination that none exists.

Running time: $O(|V(G)|^{18})$.

Proof. By 7.4 and 7.5 in time $O(|V(G)|^{18})$ we can produce a type I or a type II coloring of G , if one exists. Hence, we may assume there does not exist a type I or a type II coloring of G . Let C be the 6-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$, and suppose (C, p) is a shell in G . Observe that such an induced subgraph can be found in time $O(|V(G)|^7)$. Since G is clean, by 7.6.3, it follows that S_p^C is empty, and so we may partition $V(G) = V(C) \cup \{p\} \cup Q_p^C \cup R_p^C$ as usual. Next, fix a 3-coloring c of $G[V(C) \cup \{p\}]$, that is not a type I coloring with respect to C . Define the order 3 palette L_c^C of G as follows: For every $v \in V(G)$, set

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \cup \{p\} \\ \{c(p)\} & , \quad \text{if } v \notin V(C) \cup \{p\}, \text{ and } v \text{ has the same anchors as } p \text{ in } C \\ \{1, 2, 3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in Q_p^C with respect to $V(C) \cup \{p\}$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus R_p^C$, while $|L_c^C(v)| = 3$ for all $v \in R_p^C$. Observe that, since G does not have a type II coloring, (G, L_c^C) is colorable if and only if the 3-coloring c of $G[V(C) \cup \{p\}]$ extends to a 3-coloring of G .

Let A' be the set of vertices in Q_p^C with a neighbor in R_p^C , and for every non-empty subset $X \subseteq \{1, 2, 3\}$, define $A'_X = \{a \in A' \text{ with } L_c^C(a) = X\}$.

(1) Let $\{i, j, k\} = \{1, 2, 3\}$. If $q_1 \in A'_{\{i,j\}}$, $q_2 \in A'_{\{j,k\}}$, and $r_1 \in R_p^C \cap (N(q_1) \setminus N(q_2))$ and $r_2 \in R_p^C \cap (N(q_2) \setminus N(q_1))$, then q_1 is adjacent to q_2 .

Suppose q_1 is non-adjacent to q_2 . Since $L(q_1) = \{i, j\}$ and $L(q_2) = \{j, k\}$, it follows that $N(q_1) \cap N(q_2) \cap (V(C) \cup \{p\})$ is empty. First, assume $q_1 \in S^C$. By symmetry, we may assume $q_1 \in S_0^C$ and $p \in P^C(\{0, 3\})$. Since G is triangle-free, it follows that q_1 is non-adjacent to p . Suppose $q_2 \in PL_p^C$. However, if r_1 is non-adjacent to r_2 , then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2$ is a P_7 in G , and if r_1 is adjacent to r_2 , then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2 - r_1$ is a C_7 in G , in both case a contradiction. Hence, $q_2 \notin PL_p^C$. By 7.6.2, it follows that $|N(q_2) \cap \{v_1, v_3, v_5\}| \geq 2$. Suppose q_2 is adjacent to v_3 . Since G is triangle-free, it follows that p is non-adjacent to q_2 . However, if r_1 is non-adjacent to r_2 , then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2$ is a P_7 in G , and if r_1 is adjacent to r_2 , then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2 - r_1$ is a C_7 in G , in both case a contradiction. Hence, q_2 is non-adjacent to v_3 , and so q_2 is complete to $\{v_1, v_5\}$. If p is non-adjacent to q_2 , then $v_4 - v_3 - p - v_0 - v_1 - q_2 - r_2$ is a P_7 in G , a contradiction. Hence, p is adjacent to q_2 . However, if r_1 is non-adjacent to r_2 , then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2$ is a P_7 in G , and if r_1 is adjacent to r_2 , then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2 - r_1$ is a C_7 in G , in both case a contradiction. By symmetry, this proves that neither q_1 nor q_2 belongs to S^C .

Next, suppose $q_1 \in P^C$. By symmetry, we may assume $q_1 \in P^C(\{0, 3\})$. Suppose first that q_2 is adjacent to v_1 . Since G is triangle-free, it follows that q_2 is non-adjacent to v_2 . However, if r_1 is non-adjacent to r_2 , then $r_1 - q_1 - v_3 - v_2 - v_1 - q_2 - r_2$ is a P_7 in G , and if r_1 is adjacent to r_2 , then $r_1 - q_1 - v_3 - v_2 - v_1 - q_2 - r_2 - r_1$ is a C_7 in G , in both case a contradiction. By symmetry, it follows that q_2 is anticomplete to $\{v_1, v_2, v_4, v_5\}$. Since $q_1 \in A'_{\{i,j\}}$ and $q_2 \in A'_{\{j,k\}}$, it follows that $q_2 \in PL_p^C$, $c(p) \neq c(v_0)$, and p is non-adjacent to q_1 . Since $|L_c^C(q_1)| = 2$, it follows that $p \notin P^C(\{0, 3\})$, and so we may assume that $p \in P^C(\{1, 4\})$. Now, if r_1 is non-adjacent to r_2 , then $r_2 - q_2 - p - v_1 - v_0 - q_1 - r_1$ is a P_7 in G , and if r_1 is adjacent to r_2 , then $r_2 - q_2 - p - v_1 - v_0 - q_1 - r_1 - r_2$ is a C_7 in G , in both cases a contradiction. By symmetry, this proves that neither q_1 nor q_2 belongs to P^C .

Since not both q_1 and q_2 are adjacent to p , by 7.6.1 and 7.6.2, we may assume $q_1 \in CL^C$ is non-adjacent to p . By symmetry, we may assume $q_1 \in CL^C(1)$. Since $r_1 - q_1 - v_0 - v_1 - p - v_4 - v_3$ is not a P_7 in G , it follows that $p \notin P^C(\{1, 4\})$. And so, we may assume $p \in P^C(\{0, 3\})$. Suppose $q_2 \in CL^C$ also. Since $N(q_1) \cap N(q_2) \cap V(C)$ is empty, we may assume that $q_2 \in CL^C(0) \cup CL^C(4)$. Suppose $q_2 \in CL^C(0)$. Let C' be the 6-gon in G given by $q_2 - v_1 - v_2 - v_3 - v_4 - v_5 - q_2$. Then $r_2 \in M^{C'}(0)$ and $q_1 \in M^{C'}(2)$, contrary to 7.6.2.

Hence, $q_2 \in CL^C(4)$. However, if r_1 is non-adjacent to r_2 , then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2$ is a P_7 in G , and if r_1 is adjacent to r_2 , then $r_1 - q_1 - v_0 - p - v_3 - q_2 - r_2 - r_1$ is a C_7 in G , in both case a contradiction. Hence, $q_2 \notin CL^C$. By 7.6.1 and 7.6.2, it follows $q_2 \in PL_p^C$. However, if r_1 is non-adjacent to r_2 , then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2$ is a P_7 in G , and if r_1 is adjacent to r_2 , then $r_1 - q_1 - v_2 - v_3 - p - q_2 - r_2 - r_1$ is a C_7 in G , in both case a contradiction. This proves (1).

(2) For all distinct $i, j \in \{1, 2, 3\}$ some vertex of $V(C) \cup \{p\}$ is complete to $A'_{\{i,j\}}$.

If $A'_{\{i,j\}} = \emptyset$, then (2) trivially holds. Thus, we may assume $A'_{\{i,j\}} \neq \emptyset$. Let $\{i, j, k\} = \{1, 2, 3\}$ and define K to be the set of vertices of $V(C) \cup \{p\}$ with a neighbor in $A'_{\{i,j\}}$. Since we have updated, it follows that $c(v) = k$ for every $v \in K$. Since $G[V(C) \cup \{p\}]$ has no stable set of size 4, it follows that $|K| \leq 3$. If $|K| = 1$, then, by definition, the unique vertex of K is complete to $A'_{\{i,j\}}$. Hence, we may assume that $|K| \geq 2$. If $|K| = 2$, then, by 7.6.2, either $p \in K$ is complete to $A'_{\{i,j\}}$, or $p \notin K$ and then K is complete to $A'_{\{i,j\}}$. In either case, (2) holds. And so we may assume $|K| = 3$. By 7.6.1 and 7.6.2, it follows that $A'_{\{i,j\}} \subseteq CL^C \cup P^C \cup S^C \cup PL_p^C$. First, suppose $p \notin K$. It follows that $PL_p^C \cap A'_{\{i,j\}}$ is empty. By symmetry, we may assume $K = \{v_0, v_2, v_4\}$. Suppose further that v_0 and v_2 are not complete to $A'_{\{i,j\}}$. By (1), it follows that there exists $c_3 \in A'_{\{i,j\}} \cap CL^C(3)$, and $c_5 \in A'_{\{i,j\}} \cap CL^C(5)$. Since G is triangle-free, it follows that c_3 is non-adjacent to c_5 . By definition, there exists $r_3, r_5 \in R_p^C$ such that c_3 is adjacent to r_3 , and c_5 is adjacent to r_5 . By 7.3.6, it follows that c_3 is non-adjacent to r_5 , and c_5 is non-adjacent to r_3 . Let C' be the 6-gon in G given by $v_0 - v_1 - v_2 - c_3 - v_4 - c_5 - v_0$. Then $r_3 \in M^{C'}(3)$ and $r_5 \in M^{C'}(5)$, contrary to 7.3.5. And so, it follows that $p \in K$.

Without loss of generality, we may assume $p \in P^C(\{0, 3\})$. Let $\{a, b\} = K \cap V(C)$. By symmetry, we may assume $a = v_1$ and $b \in \{v_4, v_5\}$. We may also assume p is not complete to $A'_{\{i,j\}}$, as otherwise (2) holds immediately. By 7.6.1 and 7.6.2, it follows that there exists $q \in A'_{\{i,j\}}$ complete to $\{a, b\}$ and non-adjacent to p . We may also assume that a is not complete to $A'_{\{i,j\}}$, as otherwise (2) holds immediately. And so, by 7.6.2, there exists $q' \in PL_p^C \cap A'_{\{i,j\}}$. By definition, there exists $r, r' \in R_p^C$ such that r is adjacent to q , and r' is adjacent to q' . Suppose r is adjacent to q' . Since G is triangle-free, it follows that q is non-adjacent to q' . But now $v_2 - v_1 - q - r - q' - p - v_3 - v_2$ is a C_7 in G , a contradiction. Hence, it follows that r is non-adjacent to q' , and, by symmetry, that r' is non-adjacent to q . Suppose q is non-adjacent to q' . If r is non-adjacent to r' , then $r - q - v_1 - v_0 - p - q' - r'$ is a P_7 in G , and if r is adjacent to r' , then $r - q - v_1 - v_0 - p - q' - r' - r$ is a C_7 in G , in both cases a contradiction. Hence, q is adjacent to q' . Let C''' be the 6-gon in G given by $v_1 - v_2 - v_3 - p - q' - q - v_1$. If q is adjacent to v_5 (and therefore not to v_4), then $r' - q' - q - v_1 - v_2 - v_3 - v_4$ is a P_7 in G , a contradiction. Hence, it follows that q is non-adjacent to v_5 , that is, that $b = v_4$. Since c is not a type I coloring with respect to C , and since $c(v_1) = c(v_4) = c(p)$, it follows that $c(v_0) = c(v_2)$, and $c(v_3) = c(v_5)$. Applying the fact that G admits no type I coloring to the 6-gon $v_1 - v_2 - v_3 - p - q' - q - v_1$, we deduce

that in every coloring c' of (G, L_c^C) , $c'(q) = c(v_2)$ and $c'(q') = c(v_3)$. However, applying the fact that G admits no type I coloring with respect to the 6-gon $v_4 - v_5 - v_0 - p - q' - q - v_4$, we deduce that in every coloring c' of (G, L_c^C) , $c'(q) = c(v_5)$ and $c'(q') = c(v_0)$. But this implies that $c(v_0) = c(v_2) = c(v_5)$, a contradiction. This proves (2).

By 7.6.4, it follows that every component of R_p^C has at most two vertices. And so, by 7.6.1, 7.6.2, (1) and (2), we can apply 3.4 with

- $\tilde{A} = A'$
- $\tilde{B} = V(C) \cup \{p\} \cup (Q_p^C \setminus A')$,
- $\tilde{C} = R_p^C$, and
- $Z = \emptyset$.

Let \mathcal{P}_c be the restriction of (G, L_c^C) of size $O(|V(G)|^7)$ thus obtained, and let \mathcal{P} be the union of \mathcal{P}_c taken over all 3-colorings c of $G[V(C) \cup \{p\}]$ that are not type I colorings. By 3.4, and since there are at most 7^3 3-colorings of $G[V(C) \cup \{p\}]$, it follows that \mathcal{P} can be computed in time $O(|V(G)|^7)$. By 3.4(c), we have that (G, L_c^C) is colorable if and only if \mathcal{P}_c is colorable. Since every 3-coloring of G restricts to a 3-coloring of $G[V(C) \cup \{p\}]$, it follows that G is 3-colorable if and only if \mathcal{P} is colorable.

Consider $(G', L', X') \in \mathcal{P}$. Then $(G', L', X') \in \mathcal{P}_c$ for some coloring c of $G[V(C) \cup \{p\}]$, and c is not a type I or a type II coloring. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus R_p^C$, by 3.4(b), it follows that $|L'(v)| \leq 2$ for all $v \in V(G')$. Thus, since by 3.4 X' has size $O(|V(G)|)$, applying 1.7, we can test in time $O(|V(G)|^3)$ if (G', L', X') is colorable. Therefore, via $O(|V(G)|^7)$ applications of 1.7, we can determine if \mathcal{P} is colorable and extend any coloring of a colorable $(G', L', X') \in \mathcal{P}$ to a coloring of G in linear time. Consequently, in time $O(|V(G)|^{10})$ we can determine if \mathcal{P} is colorable. This proves 7.7. □

8 5-gons

In this section we show that if a $\{P_7, C_3, C_7, \text{shell}\}$ -free graph contains a 5-gon, then in polynomial time we can decide if the graph is 3-colorable, and give a coloring if one exists.

Let C be a 5-gon in a graph G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. We say that a vertex $v \in V(G) \setminus V(C)$ is:

- a *leaf at i* , if $N(v) \cap V(C) = \{v_i\}$ for some $i \in \{0, 1, \dots, 4\}$,
- a *clone at i* , if $N(v) \cap V(C) = \{v_{i-1}, v_{i+1}\}$ for some $i \in \{0, 1, \dots, 4\}$, where all indices are mod 5.

The following shows how we can partition the vertices of G based on their anchors in C .

8.1. Let G be a triangle-free graph, and suppose C is a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. If $v \in V(G) \setminus V(C)$, then for some $i \in \{0, 1, \dots, 4\}$, either:

1. v is a leaf at i ,
2. v is a clone at i , or
3. v is anticomplete to $V(C)$.

Proof. Consider a vertex $v \in V(G) \setminus V(C)$. If v is anticomplete to $V(C)$, then 8.1.3 holds. Thus, we may assume $N(v) \cap V(C) \neq \emptyset$, and, by symmetry, suppose that $v_0 \in N(v) \cap V(C)$. If $|N(v) \cap V(C)| = 1$, then 8.1.1 holds, and so we may assume $|N(v) \cap V(C)| \geq 2$. Since G is triangle-free, it follows that v is anticomplete to $\{v_1, v_4\}$. Since G is triangle-free, it follows that v is mixed on $\{v_2, v_3\}$ and so 8.1.2 holds. This proves 8.1. \square

Let G be a triangle-free graph. Suppose C is a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. Using 8.1 we partition $V(G) \setminus V(C)$ as follows:

- Let $M^C(i)$ be the set of leaves at i and define $M^C = \bigcup_{i=0}^4 M^C(i)$.
- Let $CL^C(i)$ be the set of clones at i and define $CL^C = \bigcup_{i=0}^4 CL^C(i)$.
- Let A^C be the set of vertices anticomplete to $V(C)$.

By 8.1, it follows that $V(G) = V(C) \cup M^C \cup CL^C \cup A^C$. Furthermore, we partition $A^C = X^C \cup Y^C \cup Z^C$, where

- X^C is the set of vertices in A^C with a neighbor in M^C ,
- Y^C is the set of vertices in $A^C \setminus X^C$ with a neighbor in CL^C , and
- $Z^C = A^C \setminus (X^C \cup Y^C)$.

Finally, we define subsets of X^C , Y^C and M^C , for every $i \in \{0, \dots, 4\}$ as follows:

- Let $X^C(i)$ be the set of vertices of X^C with a neighbor in $M^C(i)$.
- Let $Y^C(i)$ be the set of vertices of Y^C with a neighbor in $CL^C(i)$.
- Let M_i^C be the set of vertices of M^C with a neighbor in $X^C(i)$.

And so, for a given 5-gon C in time $O(|V(G)|^2)$ we obtain the partition $V(C) \cup M^C \cup CL^C \cup X^C \cup Y^C \cup Z^C$ of $V(G)$. Now, we establish several properties of this partition. By definition and 8.1, it follows that:

8.2. *If G is a triangle-free graph, then for every 5-gon C in G the following hold:*

1. *Every vertex in X^C has a neighbor in M^C .*
2. *Every vertex in Y^C has a neighbor in CL^C .*
3. *Y^C is anticomplete to M^C .*

Recall that for a fixed subset X of $V(G)$, we say a vertex $v \in V(G) \setminus X$ is *mixed on an edge of X* , if there exist adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$.

8.3. *If G is a $\{P_7, C_3\}$ -free graph, then for every 5-gon C in G the following hold:*

1. *No vertex in M^C is mixed on an edge of A^C .*
2. *X^C is stable and anticomplete to $Y^C \cup Z^C$.*
3. *Both $M^C(i)$ and $CL^C(i)$ are stable for every $i \in \{0, \dots, 4\}$.*

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$. Suppose for adjacent $a, a' \in A^C$, there exists $m \in M^C$ which is adjacent to a' and non-adjacent to a . By symmetry, we may assume $m \in M^C(0)$. However, then $a - a' - m - v_0 - v_1 - v_2 - v_3$ is a P_7 in G , a contradiction. This proves 8.3.1.

Consider a vertex $x \in X^C$. By 8.2.1, there exists $m \in M^C$ adjacent to x . If there exists $x' \in N(x) \cap A^C$, then, since G is triangle-free m is non-adjacent to x' , and it follows that m is mixed on an edge of A^C , contradicting 8.3.1. This proves 8.3.2.

For every $i \in \{0, \dots, 4\}$, by definition v_i is complete to $M^C(i)$ and v_{i+1} is complete to $CL^C(i)$, where all indices are mod 5. Since G is triangle-free, it follows that 8.3.3 holds. This proves 8.3. □

8.4. *Let G be a clean, connected $\{P_7, C_3, C_7, \text{shell}\}$ -free graph. Then for every 5-gon C in G and $i \in \{0, \dots, 4\}$ the following hold:*

1. *$X^C(i)$ is anticomplete to $M^C \setminus M^C(i)$; in other words, $M_i^C \subseteq M^C(i)$.*
2. *$X^C(0) \cup \dots \cup X^C(4)$ gives a partition of X^C .*
3. *Every vertex in $X^C(i)$ has a neighbor in $CL^C(i)$.*
4. *$X^C(i)$ is anticomplete to $CL^C \setminus CL^C(i)$.*
5. *$Y^C(i)$ is anticomplete to $CL^C(i+1) \cup CL^C(i-1)$, where all indices are mod 5.*
6. *M_i^C is anticomplete to $V(G) \setminus (M_i^C \cup CL^C(i) \cup X^C(i))$.*

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$. It is enough to prove this statement for $i = 0$. Let $x \in X^C(0)$, and let $m \in M^C(0)$ be adjacent to x . Then $x \in M_0^C$. By 8.3.2, X^C is stable and anticomplete to $Y^C \cup Z^C$, and so it follows that $N(x) \subseteq M^C \cup CL^C$. Suppose there exists $m' \in N(x) \cap (M^C \setminus M_0^C)$. Since G is triangle-free, m is non-adjacent to m' . By symmetry, we may assume $m' \in M^C(3) \cup M^C(4)$. However, if $m' \in M^C(4)$, then $m' - x - m - v_0 - v_1 - v_2 - v_3$ is a P_7 in G , and if $m' \in M^C(3)$, then $m - x - m' - v_3 - v_2 - v_1 - v_0 - m$ is a C_7 in G , in both cases, a contradiction. Hence, x is anticomplete to $M^C \setminus M_0^C$. This proves 8.4.1, which, by 8.2.1, immediately implies 8.4.2. Since v_0 is complete to $M^C(0)$ and G has no dominated vertices, it follows that there exists $c \in CL^C \setminus (CL^C(1) \cup CL^C(4))$ adjacent to x . Since G is triangle-free, c is non-adjacent to m . Suppose $c \notin CL^C(0)$. By symmetry, we may assume $c \in CL^C(2)$. However, then $v_0 - m - x - c - v_3 - v_4 - v_0$ with v_1 is a shell in G , a contradiction. Hence, x has a neighbor in $CL^C(0)$. This proves 8.4.3. Now, we prove 8.4.4 and 8.4.5. We have already shown that $X^C(0)$ is anticomplete to $CL^C(2) \cup CL^C(3)$. Let $c \in CL^C(0)$ be adjacent to $z \in X^C(0) \cup Y^C(0)$. Suppose there exists $c' \in CL^C(1) \cup CL^C(4)$ adjacent to z . By symmetry, we may assume $c' \in CL^C(1)$. Since G is triangle-free, c' is non-adjacent to c . However, then $c - z - c' - v_2 - v_3 - v_4 - c$ with v_1 is a shell in G , a contradiction. Hence, $X^C(0)$ is anticomplete to $CL^C \setminus CL^C(0)$, and $Y^C(0)$ is anticomplete to $CL^C(1) \cup CL^C(4)$. This proves 8.4.4 and 8.4.5.

Next we prove 8.4.6. Recall m is an arbitrary vertex of $M^C(0)$, and that $x \in X^C(0)$ is adjacent to m and to $c \in CL^C(0)$. By definition, 8.2.3 and 8.4.1, it follows that M_0^C anticomplete to $(X^C \setminus X^C(0)) \cup Y^C \cup Z^C$. Suppose there exists $m' \in M^C \setminus M_0^C$ adjacent to m . By 8.3.3, it follows that $M^C(0)$, and thus M_0^C , is stable, and so it follows that $m' \in M^C \setminus M^C(0)$. Since G is triangle-free, x is non-adjacent to m' . By symmetry, we may assume $m' \in M^C(1) \cup M^C(2)$. However, if $m' \in M^C(1)$, then $x - m - m' - v_1 - v_2 - v_3 - v_4$ is a P_7 in G , and if $m' \in M^C(2)$, then $m - m' - v_2 - v_3 - v_4 - v_0 - m$ with v_1 is a shell in G , in both cases, a contradiction. Hence, M_0^C is anticomplete to $M^C \setminus M_0^C$. Finally, we show that M_0^C is anticomplete to $CL^C \setminus CL^C(0)$. Since v_0 is complete to $M^C(0)$ and G is triangle-free, it follows that M_0^C is anticomplete to $CL^C(1) \cup CL^C(4)$. Suppose there exists $c'' \in CL^C(2) \cup CL^C(3)$ adjacent to m . Since G is triangle-free, c'' is anticomplete to $\{c, x\}$. By symmetry, we may assume $c'' \in CL^C(2)$. However, then $x - m - c'' - v_3 - v_4 - c - x$ with v_0 is a shell in G , a contradiction. Hence, M_0^C is anticomplete to $CL^C \setminus CL^C(0)$. This proves 8.4.6. □

8.5. *Let G be a clean, connected $\{P_7, C_3, C_7, S_7\}$ -free graph. Then for every 5-gon C in G the following hold:*

1. Z^C is empty.
2. Every component of Y^C has size two.
3. $Y^C(0), \dots, Y^C(4)$ are pairwise disjoint and anticomplete to each other.

4. Every component of $M_i^C \cup X^C(i) \cup Y^C(i)$ has size two for every $i \in \{0, \dots, 4\}$.

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$.

(1) For every $c \in CL^C$, there exists $b_1, b_2, b_3 \in V(C)$ such that $b_1 - b_2 - b_3 - c$ is an induced path.

Consider a vertex $c \in CL^C$. By symmetry, we may assume $c \in CL^C(0)$. And so, $v_3 - v_2 - v_1 - c$ is the desired induced path. This proves (1).

By 8.2, 8.3.2 and (1), we may apply 5.1 letting $P = V(C)$, $Q = CL^C$, $R = Y^C$, $S = Z^C$, and $T = M^C \cup X^C$. It thus follows that 8.5.1 holds and that every component of Y^C has size at most two. Now, suppose $y \in Y^C$ is a singleton component of Y^C . By 8.2.2, there exists $c \in CL^C$ adjacent to y . By symmetry, we may assume $c \in CL^C(0)$, and so $y \in Y^C(0)$. By 8.4.5, y is anticomplete to $CL^C(1) \cup CL^C(4)$, and so, by 8.2.3, it follows that y is anticomplete to $V(G) \setminus (CL^C(0) \cup CL^C(2) \cup CL^C(3))$. Since v_1 does not dominate y , it follows that y has a neighbor in $CL^C(3)$. Since v_4 does not dominate y , it follows that y has a neighbor in $CL^C(2)$. However, then y has a neighbor in $CL^C(2)$ and in $CL^C(3)$, contrary to 8.4.5. This proves 8.5.2.

(2) If $\{y, y'\}$ is the vertex set of a component of Y^C , then there exists a unique $i \in \{0, \dots, 4\}$ such that vertices of $CL^C(i)$ have neighbors in $\{y, y'\}$.

Suppose $\{y, y'\}$ is the vertex set of a component of Y^C . By 8.2.2, there exists $c \in CL^C$ adjacent to y . Since G is triangle-free, y' is non-adjacent to c . By symmetry, we may assume $c \in CL^C(0)$. Let C' be the 5-gon given by $c - v_1 - v_2 - v_3 - v_4 - c$. It follows that $y \in M^{C'}(0)$, $y' \in X^{C'}(0)$ and, since G is triangle-free, $CL^C(j) = CL^{C'}(j)$ for $j = 0, 2, 3$. And so, by 8.4.4 applied to C' , it follows that y' has a neighbor in $CL^{C'}(0) = CL^C(0)$ and is anticomplete to $CL^C(2) \cup CL^C(3)$. In particular, $y' \in Y^C(0)$. By 8.4.5 applied to C , it follows that y' is anticomplete to $CL^C(1) \cup CL^C(4)$. And so, it follows that y' is anticomplete to $CL^C \setminus CL^C(0)$. But now reversing the roles of y and y' , it follows that y is anticomplete to $CL^C \setminus CL^C(0)$. This proves (2).

By 8.2.2, every vertex $y \in Y^C$ has a neighbor in CL^C and so (2) implies that $Y^C(0), \dots, Y^C(4)$ are pairwise disjoint and anticomplete to each other. This proves 8.5.3.

(3) Every component of $M_i^C \cup X^C(i)$ has size two for every $i \in \{0, \dots, 4\}$.

We may assume $i = 0$. By 8.3.3, it follows that $M^C(0)$, and thus M_0^C , is stable. By definition, 8.4.3 and 8.4.4, it follows that every vertex in $M_0^C \cup X^C(0)$ has a neighbor in $CL^C(0) \cup \{v_0\}$. By 8.3.2, 8.4.1, 8.4.2 and 8.4.4, it follows that $X^C(0)$ is anticomplete to $V(G) \setminus (X^C(0) \cup M_0^C \cup CL^C(0))$. By 8.4.6, it follows that M_0^C is anticomplete to $V(G) \setminus (M_0^C \cup CL^C(0) \cup X^C(0))$. Since $v_0 - v_1 - v_2 - v_3$ is an induced path, by (1), we

may apply 5.1 letting $P = V(C) \setminus \{v_0\}$, $Q = CL^C(0) \cup \{v_0\}$, $R = M_0^C \cup X^C(0)$, $S = \emptyset$, and $T = V(G) \setminus (V(C) \cup M_0^C \cup CL^C(0) \cup X^C(0))$. Hence, every component of $M_0^C \cup X^C$ has size at most two. However, since every vertex in M_0^C has a neighbor in $X^C(0)$ and every vertex in $X^C(0)$ has a neighbor in M_0^C , it follows that (3) holds.

By 8.2.3 and 8.3.2, it follows that $M^C \cup X^C$ is anticomplete to Y^C . Hence, together 8.5.2 and (3) imply that 8.5.4 holds. This proves 8.5. □

Now, we prove the main result of the section.

8.6. *There is an algorithm with the following specifications:*

Input: *A clean, connected $\{P_7, C_3, C_7, \text{shell}\}$ -free graph G which contains a 5-gon.*

Output: *A 3-coloring of G , or a determination that none exists.*

Running time: $O(|V(G)|^6)$.

Proof. Let C be a 5-gon in G given by $v_0 - v_1 - v_2 - v_3 - v_4 - v_0$; clearly C can be found in time $O(|V(G)|^5)$. In time $O(|V(G)|^2)$, we can partition $V(G) = V(C) \cup CL^C \cup M^C \cup X^C \cup Y^C \cup Z^C$ as well as determine $X^C(i), Y^C(i)$ and M_i^C for every $i \in \{0, \dots, 4\}$. Since G is clean, by 8.5.1, it follows that Z^C is empty and, by 8.4.2 and 8.5.3, we obtain the partitions $X^C(0) \cup \dots \cup X^C(4)$ of X^C and $Y^C(0) \cup \dots \cup Y^C(4)$ of Y^C . Next, fix a 3-coloring c of $G[V(C)]$. By symmetry, we may assume $c(v_1) = c(v_3)$ and $c(v_2) = c(v_4)$. Define the order 3 palette L_c^C of G as follows:

$$L_c^C(v) = \begin{cases} \{c(v)\} & , \quad \text{if } v \in V(C) \\ \{1, 2, 3\} & , \quad \text{otherwise} \end{cases}$$

Next, update the vertices in $CL^C \cup M^C$ with respect to $V(C)$. And so, $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C \cup Y^C)$. Furthermore, $|L_c^C(v)| = 2$ if and only if $v \in M^C \cup CL^C(2) \cup CL^C(3)$. Now, update the vertices in $X^C \cup Y^C$ with respect to $CL^C \cup M^C$. By 8.4.3, it follows that $|L_c^C(v)| = 3$ if and only if $v \in X^C(2) \cup Y^C(2) \cup X^C(3) \cup Y^C(3)$. For every $j \in \{2, 3\}$, by 8.4.3, every vertex of $M_j \cup X^C(j) \cup Y^C(j)$ has a neighbor in $CL^C(j) \cup \{v_j\}$ and, by 8.5.4, every component of $M_j \cup X^C(j) \cup Y^C(j)$ has size 2. Let \mathcal{L}_1 be the set of $O(|V(G)|^2)$ subpalettes of L_c^C obtained from 2.3 applied with

- $x = v_1$,
- $S = CL^C(2) \cup \{v_2\}$,
- $\hat{A} \cup \hat{B} = M_2^C \cup X^C(2) \cup Y^C(2)$,
- $Y = V(G) \setminus (\{v_1, v_2\} \cup CL^C(2) \cup M_2 \cup X^C(2) \cup Y^C(2))$, and

- $X = \emptyset$.

Next, for every $L \in \mathcal{L}_1$, let $\mathcal{L}(L)$ be the set of $O(|V(G)|^2)$ subpalettes of L obtained from 2.3 applied with

- $x = v_4$,
- $S = CL^C(3) \cup \{v_3\}$,
- $\hat{A} \cup \hat{B} = M_3^C \cup X^C(3) \cup Y^C(3)$, and
- $Y = V(G) \setminus (\{v_3, v_4\} \cup CL^C(3) \cup M_3 \cup X^C(3) \cup Y^C(3))$, and
- $X = \emptyset$.

Finally, let $\mathcal{L}_c = \{\mathcal{L}(L) : L \in \mathcal{L}_1\}$ be the set of $O(|V(G)|^4)$ subpalettes of L_c^C thus obtained. By 2.3, \mathcal{L}_c can be computed in time $O(|V(G)|^6)$. Since $X = \emptyset$, by 2.3(b), we have that (G, L_c^C) is colorable if and only if (G, \mathcal{L}_c) is colorable. Let \mathcal{L} be the union of the sets \mathcal{L}_c taken over all 3-colorings c of $G[V(C)]$. Then G is 3-colorable if and only if (G, \mathcal{L}) is 3-colorable. Since $|L_c^C(v)| \leq 2$ for all $v \in V(G) \setminus (X^C(2) \cup Y^C(2) \cup X^C(3) \cup Y^C(3))$, by 2.3(a), it follows that $|L(v)| \leq 2$ for all $L \in \mathcal{L}$ and $v \in V(G)$. Thus, by 1.6 we can test in time $O(|V(G)|^2)$ if (G, L) is colorable for every $L \in \mathcal{L}$. Since there are at most 5^3 3-colorings of $G[V(C)]$, it follows that \mathcal{L} consists of $O(|V(G)|^4)$ subpalettes of L_c^C , and so, via $O(|V(G)|^4)$ applications of 1.6, we can determine if (G, \mathcal{L}) is colorable. That is, in time $O(|V(G)|^6)$ we can determine if there exists a 3-coloring c of $G[V(C)]$ that extends to a 3-coloring of G , and give an explicit 3-coloring c' of G such that $c'(v) = c(v)$ for all $v \in V(C)$, if one exists. Since every 3-coloring of G restricts to a 3-coloring of $G[V(C)]$, this proves 8.6. □

9 Main Result

In this section we prove the main result of this paper 1.8, which we restate:

9.1. *There is an algorithm with the following specifications:*

Input: A $\{P_7, C_3\}$ -free graph G .

Output: A 3-coloring of G , or a determination that none exists.

Running time: $O(|V(G)|^{18})$.

Proof. By 4.3, at the expense of carrying out a time $O(|V(G)|^5)$ procedure we may assume G is clean. Via breadth-first search in time $O(|V(G)|^2)$ we can determine the components of G , and so we may also assume G is connected. By 6.6, if G contains a 7-gon, then in time $O(|V(G)|^{10})$ we can either produce a 3-coloring of G , or determine that none exists. Hence, we may assume G is a $\{P_7, C_3, C_7\}$ -free graph. By 7.7, if G contains a shell, then in time $O(|V(G)|^{18})$ we can either produce a 3-coloring of G , or determine that none exists. Hence, we may assume G is a $\{P_7, C_3, C_7, \text{shell}\}$ -free graph. By 8.6, if G contains a 5-gon, then in time $O(|V(G)|^6)$ we can either produce a 3-coloring of G , or determine that none exists. Hence, we may assume G is a $\{P_7, C_3, C_5, C_7, \text{shell}\}$ -free graph. Since G is P_7 -free, it follows that G is C_k -free for all $k > 7$. And so, G is bipartite and in time $O(|V(G)|^3)$ we can produce a 2-coloring of G . This proves 9.1. □

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References

- [1] B. Aspvall, M. Plass, R. Tarjan, A Linear-Time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas, *Inf. Process. Lett.* 8 (3) (1979), 121-123.
- [2] M. Chudnovsky, P. Maceli and M. Zhong, Three-coloring graphs with no induced seven-vertex path II : using a triangle, in preparation.
- [3] K. Edwards, The complexity of colouring problems on dense graphs, *Theoret. Comput. Sci.* 43 (1986), 337-343.
- [4] I. Holyer, The NP-completeness of edge coloring, *SIAM J. Comput.* 10 (1981), 718-720.
- [5] C.T. Hoàng, M. Kamiński, V.V. Lozin, J. Sawada and X. Shu, Deciding k -colorability of P_5 -free graphs in polynomial time, *Algorithmica* 57 (2010), 74-81.
- [6] S. Huang, Improved Complexity Results on k -Coloring P_t -Free Graphs, *Proc. MFCS 2013, LNCS*, to appear.
- [7] R. M. Karp, Reducibility Among Combinatorial Problems, *Complexity of Computer Computations*, New York: Plenum., 85-103.

- [8] A. King and B. Reed, Bounding χ in Terms of ω and Δ for Quasi-Line Graphs, Journal of Graph Theory, Vol. 59 (2008), 215-228.
- [9] M. Kamiński and V.V. Lozin, Coloring edges and vertices of graphs without short or long cycles. Contrib. Discrete. Math. 2 (2007), 61-66.
- [10] D. Leven and Z. Galil, NP-completeness of finding the chromatic index of regular graphs. J. Algorithm 4 (1983), 35-44.
- [11] B. Randerath and I. Schiermeyer, 3-Colorability $\in P$ for P_6 -free graphs, Discrete Appl. Math. 136 (2004), 299-313.
- [12] J. Stacho, private communication.
- [13] L. Stockmeyer, Planar 3-colorability is polynomial complete, SIGACT News (1973), 1925.